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Generalized Top and Bottom Binary *n*-Tuples

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Abstract

A complex stochastic Boolean system (CSBS) depends on an arbitrary number n of random Boolean variables. The behavior of a CSBS is determined by the ordering between the occurrence probabilities $\Pr\{u\}$ of the 2^n associated binary strings $u \in \{0,1\}^n$. In this context, a binary n-tuple is called top (bottom, respectively) if its occurrence probability is always among the 2^{n-1} largest (smallest, respectively) ones. In this paper we generalize these n-tuples by defining and studying the k-top and k-bottom binary n-tuples, i.e., those whose occurrence probabilities are always among the k largest (smallest, respectively) ones ($1 \leq k \leq 2^n$). These results can be applied to the reliability analysis of many different technical systems, arising from diverse fields of Engineering.

Keywords: complex stochastic Boolean systems, reliability engineering, complementary binary n-tuples, intrinsic order, intrinsic order graph, top binary n-tuples, bottom binary n-tuples, generalized top binary n-tuples, generalized bottom binary n-tuples.

1 Introduction

In this paper, we analyze the behavior of those complex systems depending on an arbitrary number n of random Boolean variables. That is, the n basic variables of the system are assumed to be stochastic and they only take two possible values: either 0 or 1. We call such a system a complex stochastic Boolean system (CSBS). Each one of the 2^n possible elementary states associated to a CSBS is given by a binary n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ of 0s and 1s, and it has its own occurrence probability $\Pr\{(u_1, \ldots, u_n)\}$.

In probability theory and statistics, a Bernoulli variable is a discrete random variable which takes value 1 with (success) probability p and value 0 with (failure) probability $1-p$, and p is called the parameter of the Bernoulli distribution $(0<p<1)$. Hence, a CSBS on n variables $x_1,\ldots,x_n$ can be modeled by the n-dimensional Bernoulli distribution with parameters $p_1,\ldots,p_n$ defined by

$$\Pr\{x_i=1\}=p_i,\ \Pr\{x_i=0\}=1-p_i,$$

Throughout this paper we assume that the n Bernoulli variables x_i are statistically independent, so that the occurrence probability of a given binary string u of length n can be easily computed as

$$\Pr\{\,u\}=\prod_{i=1}^{n}p_i^{u_i}\left(1-p_i\right)^{1-u_i}\quad\text{for all } u\in\{0,1\}^n, \tag{1}$$

that is, $\Pr\{\,u\}$ is the product of factors p_i if $u_i=1$, $1-p_i$ if $u_i=0$ [1].

Example 1. Let $n=4,\ u=(0,1,0,1)$ and $p_1=0.1,\ p_2=0.2,\ p_3=0.3,\ p_4=0.4$. Using Equation (1) we have

$$\Pr\{(0,1,0,1)\}=(1-p_1)\,p_2\,(1-p_3)\,p_4=0.0504.$$

The behavior of a CSBS is determined by the ordering between the current values of the 2^n associated binary n-tuple probabilities $\Pr\{u\}$. Computing all these 2^n probabilities –using Equation (1)– and ordering them in decreasing or increasing order of their values is only possible in practice when the number n of basic variables is small. For large values of n, it is necessary to use alternative procedures to compare the binary string probabilities. For this purpose, in in [2, 3] we have established a simple positional criterion that allows one to compare two given elementary state probabilities, $\Pr\{u\},\Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the n-tuples u,v. We have called it the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it is exclusively determined by the positions of the 0s and 1s in the binary strings.

The only two required assumptions for applying the intrinsic order criterion to a given CSBS are the following: the n marginal Bernoulli variables of the system $x_1,\ldots,x_n$ must be mutually independent and the n corresponding Bernoulli parameters $p_1,\ldots,p_n$ must satisfy the condition

$$0<p_1\le p_2\le\ldots\le p_n\le 1/2,\quad p_i=\Pr\{x_i=1\}\ \ (1\le i\le n)\,. \tag{2}$$

Although the hypothesis (2) is essential for our theoretical results (indeed it is the basic assumption of our model), fortunately it is not restrictive for practical applications (as we shall explain in the next Section). Among the many different topics concerning the behavior of CSBSs that can be derived from the intrinsic order criterion, we focus or attention on the top, bottom and jumping binary n-tuples, defined in [4]. A binary n-tuple is called top (bottom, respectively) if its occurrence probability is "always" among the 2^{n-1} largest (smallest, respectively) ones. Here and from now on, the term "always" means for any basic probabilities $p_1,\ldots,p_n$ satisfying Equation (2). A binary n-tuple is called jumping if it is neither top, nor bottom.

In this context, the aim of this paper is to generalize the top and bottom binary n-tuples by studying the k-top and k-bottom, respectively, binary n-tuples, defined as those binary n-tuples whose occurrence probabilities are always among the k largest or among the k smallest, respectively, ones $(1 \leq k \leq 2^n)$. This new approach has both theoretical and practical interest for the study of CSBSs and, in particular, for the reliability analysis of many different technical systems arising from diverse areas of Engineering. More precisely, the failure probability of many technical systems described by stochastic Boolean functions can be estimated by selecting system elementary states with large occurrence probabilities [3, 5]. This can be performed by using the intrinsic order model [3].

For this purpose, this paper has been organized as follows. In Section 2, we present some background on the intrinsic order relation, the intrinsic order graph and on the top, bottom and jumping binary n-tuples, enabling non-specialists to follow the paper without difficulty. Section 3 is devoted to the study of the k-top and k-bottom binary n-tuples. Finally, in Section 4 we present our conclusions.

2 Background in Intrinsic Order

2.1 Intrinsic Order on $\{0,1\}^n$

Fist, we must set the following notations. Throughout this paper, the decimal numbering of a binary string u is denoted by the symbol $u_{(10}$. We use this symbol, instead of the more usual notation u_{10}, to avoid confusions with the 10-th component u_{10} of the binary string u. In the following, we use indistinctly the binary representation or the decimal representation to denote the elements of $\{0,1\}^n$, and we represent the conversion between these two numbers systems by the symbol "$\equiv$". Also, the Hamming weight of a binary n-tuple u (i.e., the number of 1-bits in u) will be denoted, as usual by $w_H(u)$, i.e.,

$$(u_1, \ldots, u_n) \equiv u_{(10} = \sum_{i=1}^{n} 2^{n-i} u_i, \quad w_H(u) = \sum_{i=1}^{n} u_i,$$

e.g., for $n = 5$ we have

$$(1,0,1,1,1) \equiv 2^0 + 2^1 + 2^2 + 2^4 = 23, \quad w_H(1,0,1,1,1) = 4.$$

According to Equation (1), the ordering between two given binary string probabilities $\Pr(u)$ and $\Pr(v)$ depends, in general, on the parameters p_i of the Bernoulli distribution, as the following simple example shows.

Example 2. Let $n = 3,\ u = (0,1,1)$ and $v = (1,0,0)$. Using Equation (1) we have

$$p_1 = 0.1,\ p_2 = 0.2,\ p_3 = 0.3 : \Pr\{(0,1,1)\} = 0.054 < \Pr\{(1,0,0)\} = 0.056,$$

$$p_1 = 0.2,\ p_2 = 0.3,\ p_3 = 0.4 : \Pr\{(0,1,1)\} = 0.096 > \Pr\{(1,0,0)\} = 0.084.$$

As mentioned in Section 1, to overcome the exponential complexity inherent to the task of computing and sorting the 2^n binary string probabilities (associated to a CSBS with n Boolean variables), we have introduced the following intrinsic order criterion [2, 3], denoted from now on by the acronym IOC.

Theorem 1 (The intrinsic order theorem) *Let $n \geq 1$. Let $x_1, \ldots, x_n$ be n mutually independent Bernoulli variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy*

$$0 < p_1 \leq p_2 \leq \ldots \leq p_n \leq 0.5. \tag{3}$$

Then the probability of the n-tuple $v = (v_1, \ldots, v_n) \in \{0,1\}^n$ is intrinsically less than or equal to the probability of the n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (3)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 1. In the following, we assume that the parameters p_i always satisfy condition (3). Note that this hypothesis is not restrictive for practical applications because, if for some $i : p_i > \frac{1}{2}$, then we only need to consider the variable $\overline{x_i} = 1 - x_i$, instead of x_i. Next, we order the n Bernoulli variables by increasing order of their probabilities.

Remark 2. The $\binom{0}{1}$ column preceding to each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

Remark 3. The term *corresponding,* used in Theorem 1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding to each other. In other words: For each $\binom{1}{0}$ column in matrix M_v^u, the number of preceding $\binom{0}{1}$ columns must be strictly greater than the number of preceding $\binom{1}{0}$ columns.

The matrix condition IOC, stated by Theorem 1, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it only depends on the relative positions of the 0s and 1s in the binary n-tuples u, v. Theorem 1 naturally leads to the following partial order relation on the set $\{0,1\}^n$ [2, 3]. The so-called intrinsic order will be denoted by "$\preceq$", and we shall write $u \succeq v$ ($u \preceq v$) to indicate that u is intrinsically greater (less) than or equal to v.

Definition 1 *For all $u, v \in \{0,1\}^n$*

$$v \preceq u \ \ iff \ \ \Pr\{v\} \leq \Pr\{u\} \textit{ for all set } \{p_i\}_{i=1}^n \ s.t. \ (3) \ \ iff \ \ M_v^u \textit{ satisfies IOC.}$$

From now on, the partially ordered set (poset, for short) $(\{0,1\}^n, \preceq)$ will be denoted by I_n.

Example 3. Let $n = 3$. Neither $3 \equiv (0,1,1) \preceq 4 \equiv (1,0,0)$, nor $4 \equiv (1,0,0) \preceq 3 \equiv (0,1,1)$ because the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

do not satisfy IOC (Remark 3). Therefore, $(0,1,1)$ and $(1,0,0)$ are incomparable by intrinsic order, i.e., the ordering between $\Pr\{(0,1,1)\}$ and $\Pr\{(1,0,0)\}$ depends on the basic probabilities p_i, as Example 2 has shown.

Example 4. Let $n = 5$. Then $24 \equiv (1,1,0,0,0) \preceq 5 \equiv (0,0,1,0,1)$ because

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

satisfies IOC (Remark 2). Therefore,

$$\Pr\{(1,1,0,0,0)\} \leq \Pr\{(0,0,1,0,1)\}, \text{ for all } 0 < p_1 \leq \ldots \leq p_5 \leq 0.5.$$

Example 5. For all $n \geq 1$, the binary n-tuples

$$\left(0, \overset{n}{\ldots}, 0\right) \equiv 0 \quad \text{and} \quad \left(1, \overset{n}{\ldots}, 1\right) \equiv 2^n - 1$$

are the maximum and minimum elements, respectively, in the poset I_n. Indeed, both matrices

$$\begin{pmatrix} 0 & \ldots & 0 \\ u_1 & \ldots & u_n \end{pmatrix} \text{ and } \begin{pmatrix} u_1 & \ldots & u_n \\ 1 & \ldots & 1 \end{pmatrix}$$

satisfy the intrinsic order criterion, since they have no $\binom{1}{0}$ columns. Hence, using Definition 1, we get

$$\left(1, \overset{n}{\ldots}, 1\right) \preceq (u_1, \ldots, u_n) \preceq \left(0, \overset{n}{\ldots}, 0\right), \text{ for all } (u_1, \ldots, u_n) \in \{0,1\}^n.$$

To finish this subsection, we must recall the two following necessary (but not sufficient) conditions for intrinsic order (see [2] for the proof).

Corollary 1 *The intrinsic order respect both, the the decimal numbering and the Hamming weight. More precisely, for all $u, v \in \{0,1\}^n$*

$$u \succeq v \Rightarrow u_{(10} \leq v_{(10} \text{ and } w_H(u) \leq w_H(v).$$

2.2 The Intrinsic Order Graph

Now, we present the graphical representation of the poset I_n. The usual representation of a poset is its Hasse diagram (see, e.g., [6] for more details about posets and Hasse diagrams). This is a directed graph (digraph, for short) whose vertices are the binary n-tuples of 0s and 1s, and whose edges connect each pair (u,v) of binary n-tuples whenever u covers v (denoted by $u \triangleright v$), that is, whenever u is intrinsically greater than v with no other elements between them, i.e.

$$u \triangleright v \quad \text{iff} \quad u \succ v \text{ and there is no } w \in \{0,1\}^n \text{ s.t. } u \succ w \succ v.$$

Moreover, according to the usual convention for Hasse diagrams, if u covers v then u is drawn above v. The Hasse diagram of the poset I_n will be also called the intrinsic order graph for n variables. For small values of n, the Hasse diagram of I_n can be constructed by direct application of the intrinsic order criterion. For large values of n in [7] we have developed an algorithm for iteratively building up the digraph of I_n, for all $n \geq 2$, from the digraph of I_1. In Figure 1 the intrinsic order graph for $n = 1, 2, 3, 4$ is shown. A graph consisting of only isolated nodes with no edges is called an empty or edgeless graph. In Figure 2 the edgeless intrinsic order graph for $n = 5, 6$ is shown.

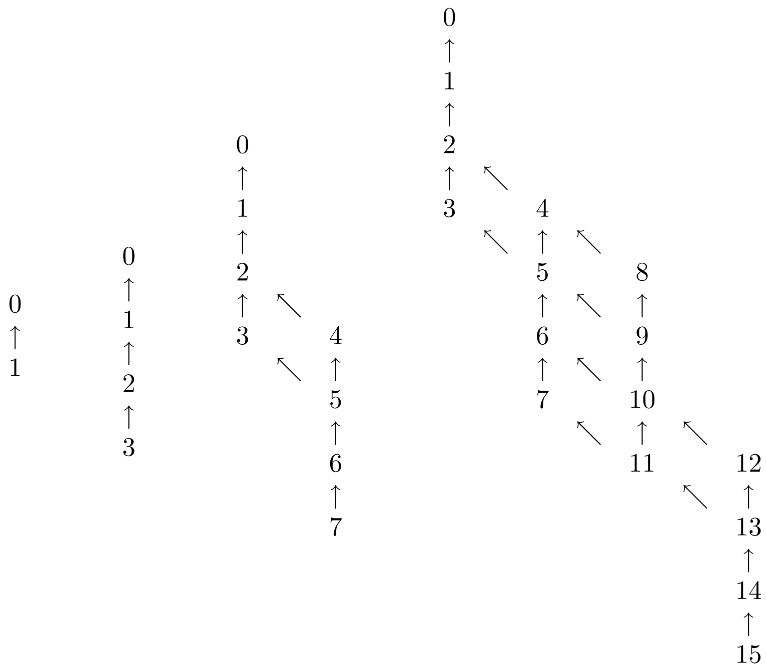

Figure 1: The intrinsic order graph for $n = 1, 2, 3, 4$.

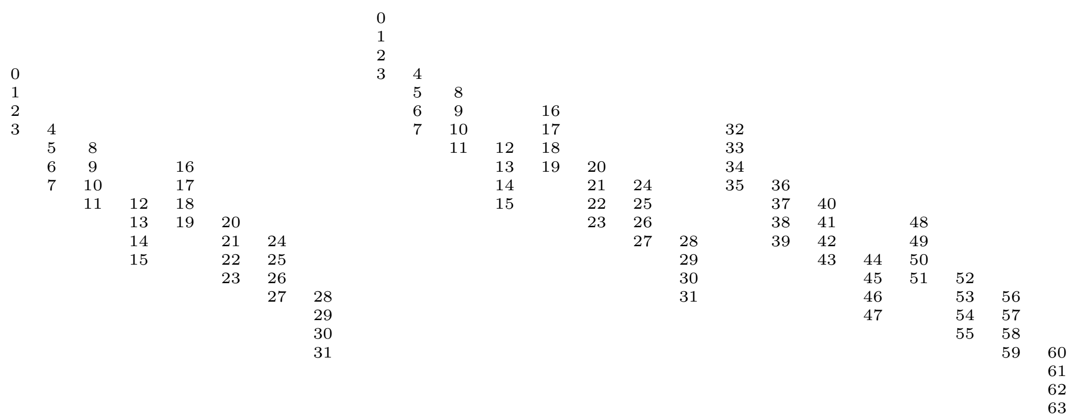

Figure 2: The edgeless intrinsic order graph for $n = 5, 6$.

2.3 Top, Bottom and Jumping binary n-tuples

The Hasse diagram of the poset I_3, i.e., the third one from the left in Figure 1, can be directly constructed by application of IOC. All pairs of binary 3-tuples are comparable by intrinsic order, with the only exception of $3 \equiv (0,1,1)$ and $4 \equiv (1,0,0)$. Indeed, Example 3 has shown that these two elements of I_3 are incomparable by intrinsic order, while Example 2 has provided us with numerical values of basic probabilities p_1, p_2, p_3 confirming this fact. Hence, assuming the hypothesis (3), if we sort the eight binary 3-tuples in decreasing order of their occurrence probabilities (downward from the largest to the smallest one), only two cases are possible. These two cases are depicted in Figure 3.

$$\begin{array}{rcl}
0 \equiv (0,0,0) & \qquad & (0,0,0) \equiv 0 \\
1 \equiv (0,0,1) & & (0,0,1) \equiv 1 \\
2 \equiv (0,1,0) & & (0,1,0) \equiv 2 \\
3 \equiv (0,1,1) & & (1,0,0) \equiv 4 \\
\cdots\cdots\cdots & & \cdots\cdots\cdots \\
4 \equiv (1,0,0) & & (0,1,1) \equiv 3 \\
5 \equiv (1,0,1) & & (1,0,1) \equiv 5 \\
6 \equiv (1,1,0) & & (1,1,0) \equiv 6 \\
7 \equiv (1,1,1) & & (1,1,1) \equiv 7
\end{array}$$

Figure 3: Top, bottom and jumping binary 3-tuples.

The left column corresponds to the case $\Pr\{(0,1,1)\} \geq \Pr\{(1,0,0)\}$, while the right column corresponds to the case $\Pr\{(0,1,1)\} \leq \Pr\{(1,0,0)\}$. Anyway, Figure 3 shows that for these two possible cases we have: (i) The 3-tuples 0, 1 and 2 are among the four first ones: they always are at the top-half of the list! (ii) The 3-tuples 5, 6 and 7 are among the four last ones: they always are at the bottom-half of the list! (iii) The 3-tuples 3 and 4 can be allocated at both positions depending on basic probabilities p_i: they jump up-down the middle line! This fact has suggested us the following nice definition [4].

Definition 2 *Let $n \geq 1$. Let the 2^n binary n-tuples be ordered in decreasing order of their occurrence probabilities. Then*

(i) The binary n-tuple u is called top if it is always among the 2^{n-1} first n-tuples $($for any set of parameters $\{p_i\}_{i=1}^n$ satisfying $(3))$.

(ii) The binary n-tuple u is called bottom if it is always among the 2^{n-1} last n-tuples $($for any set of parameters $\{p_i\}_{i=1}^n$ satisfying $(3))$.

(iii) The binary n-tuple u is called jumping if it is neither top nor bottom.

In the following, we denote by $\mathcal{T}(n), \mathcal{B}(n)$ and $\mathcal{J}(n)$ the sets of top, bottom and jumping binary n-tuples, respectively. In this way, we have the following set partition of the set of n-tuples of 0s and 1s, for all $n \geq 1$.

$$\{0,1\}^n = \mathcal{T}(n) \cup \mathcal{J}(n) \cup \mathcal{B}(n)\,. \tag{4}$$

Example 6. For $n = 3$ we have (see Figure 3)

$$\mathcal{T}(3) = \{0, 1, 2\}, \ \mathcal{J}(3) = \{3, 4\}, \ \mathcal{B}(3) = \{5, 6, 7\}.$$

The next proposition provides us with simple characterizations of the top, bottom and jumping n-tuples [4].

Theorem 2 *Let* $u = (u_1, \ldots, u_n) \in \{0, 1\}^n$. *Then*
(i) $u \in \mathcal{T}(n)$ *iff either* u *has no 1-bits, or for each 1-bit in* u*, there exists* $($*at least*$)$ *one corresponding preceding 0-bit.*
(ii) $u \in \mathcal{B}(n)$ *iff either* u *has no 0-bits, or for each 0-bit in* u*, there exists* $($*at least*$)$ *one corresponding preceding 1-bit.*
(iii) $u \in \mathcal{J}(n)$ *iff* u *contains at least one 1-bit without its corresponding preceding 0-bit and at least one 0-bit without its corresponding preceding 1-bit.*

Remark 4. The term *corresponding*, used in Theorem 2, has the same meaning that the one explained in Remark 3 for Theorem 1. In other words, Theorem 2 can be reformulated as follows: The binary n-tuple u is top (bottom, respectively) iff either u is the n-tuple $(0, \ldots, 0)$ ($(1, \ldots, 1)$, respectively), or for each 1-bit (0-bit, respectively) in u, the number of preceding 0-bits (1-bits, respectively) is strictly greater than the number of preceding 1-bits (0-bits, respectively).

To finish this Section, we describe a recursive algorithm for generating all the top n-tuples from the only top 1-tuple 0. The algorithm generates all the top n-tuples by adding one 0-bit at the end of all the Top $n-1$-tuples, and by adding one 1-bit at the end of the Top $n-1$-tuples excepting those for which $n-1$ is even and the number of 0s equals the number of 1s [4]. This algorithm is illustrated by Figure 4, where the n-th column contains all top n-tuples, for all $n \geq 1$.

$$\begin{array}{lllllllll}
 & & & & & & & \nearrow & (0,0,0,0,0)\ldots\\
 & & & & & \nearrow & (0,0,0,0) & & \\
 & & & \nearrow & (0,0,0) & & & \searrow & (0,0,0,0,1)\ldots\\
 & & & & & & & \nearrow & (0,0,0,1,0)\ldots\\
 & & & & & \searrow & (0,0,0,1) & & \\
 & \nearrow & (0,0) & & & & & \searrow & (0,0,0,1,1)\ldots\\
 & & & & & & & \nearrow & (0,0,1,0,0)\ldots\\
 & & & & & \nearrow & (0,0,1,0) & & \\
(0) & & & \searrow & (0,0,1) & & & \searrow & (0,0,1,0,1)\ldots\\
 & & & & & \searrow & (0,0,1,1) & \longrightarrow & (0,0,1,1,0)\ldots\\
 & & & & & & & \nearrow & (0,1,0,0,0)\ldots\\
 & \searrow & (0,1) & \longrightarrow & (0,1,0) & \nearrow & (0,1,0,0) & & \\
 & & & & & & & \searrow & (0,1,0,0,1)\ldots\\
 & & & & & \searrow & (0,1,0,1) & \longrightarrow & (0,1,0,1,0)\ldots
\end{array}$$

Figure 4: Generation of top n-tuples from $\mathcal{T}(1) = \{(0)\}$.

In the next section, we generalize Definition 2 by introducing an integer parameter k, such that $1 \leq k \leq 2^n$.

3 k-Top, k-Bottom and k-Jumping binary n-tuples

Definition 3 *Let $n \geq 1$ and $1 \leq k \leq 2^n$. Let the 2^n binary n-tuples be ordered in decreasing order of their occurrence probabilities. Then*

(i) The binary n-tuple u is called k-top if it is always among the k first n-tuples (for any set of parameters $\{p_i\}_{i=1}^n$ satisfying (3)).

(ii) The binary n-tuple u is called k-bottom if it is always among the k last n-tuples (for any set of parameters $\{p_i\}_{i=1}^n$ satisfying (3)).

(iii) The binary n-tuple u is called k-jumping if it is neither k-top nor k-bottom.

In the following, we denote by $\mathcal{T}^k(n), \mathcal{B}^k(n)$ and $\mathcal{J}^k(n)$ the sets of k-top, k-bottom and k-jumping binary n-tuples, respectively. In this way, we have the following decomposition (not necessarily disjoint partition) of the set of n-tuples of 0s and 1s, which generalizes the set partition (4).

$$\{0,1\}^n = \mathcal{T}^k(n) \cup \mathcal{J}^k(n) \cup \mathcal{B}^k(n).$$

Obviously, for the special case $k = 2^{n-1}$, Definition 3 becomes Definition 2. In other words, the 2^{n-1}-top, the 2^{n-1}-bottom and the 2^{n-1}-jumping binary n-tuples are the top, bottom and jumping binary n-tuples, respectively, i.e.,

$$\mathcal{T}^{2^{n-1}}(n) = \mathcal{T}(n), \quad \mathcal{J}^{2^{n-1}}(n) = \mathcal{J}(n), \quad \mathcal{B}^{2^{n-1}}(n) = \mathcal{B}(n).$$

Example 7. For $n = 3$ and $k = 6$ we have (see Figure 3)

$$\mathcal{T}^6(3) = \{0,1,2,3,4,5\}, \ \mathcal{J}^6(3) = \emptyset, \ \mathcal{B}^6(6) = \{2,3,4,5,6,7\}.$$

Figure 3 and Examples 6 & 7 suggest us a certain symmetry relation between the k-top and k-bottom n-tuples and their 0s and 1s components. The following definition and the following two lemmas formalize this fact.

Definition 4 *The complementary n-tuple u^c of a binary n-tuple u is obtained by changing its 0s by 1s and its 1s by 0s, i.e.,*

$$(u_1,\ldots,u_n)^c = (1-u_1,\ldots,1-u_n), \quad u + u^c = \left(1,\overset{n}{\ldots},1\right) \equiv 2^n - 1.$$

With this definition, we can state a basic symmetry property for the k-top and k-bottom binary n-tuples. First, we need the following auxiliary lemma.

Lemma 1 *For all $u, v \in \{0,1\}^n$*

$$\Pr\{u\} \geq \Pr\{v\} \ \Leftrightarrow \ \Pr\{u^c\} \leq \Pr\{v^c\}.$$

Proof. Using Equation (1), we get

$$\Pr\{u\} \geq \Pr\{v\} \Leftrightarrow \prod_{i=1}^{n} p_i^{u_i} (1-p_i)^{1-u_i} \geq \prod_{i=1}^{n} p_i^{v_i} (1-p_i)^{1-v_i} \Leftrightarrow$$

$$\prod_{i=1}^{n} p_i^{(1-v_i)-(1-u_i)} (1-p_i)^{v_i-u_i} \geq 1 \Leftrightarrow$$

$$\prod_{i=1}^{n} p_i^{1-u_i} (1-p_i)^{1-(1-u_i)} \leq \prod_{i=1}^{n} p_i^{1-v_i} (1-p_i)^{1-(1-v_i)} \Leftrightarrow \Pr\{u^c\} \leq \Pr\{v^c\},$$

since $1-u_i$ and $1-v_i$ are the i-th components of the complementary n-tuples u^c and v^c, respectively. □

Remark 5. Lemma 1 has the following interpretation. If we sort the 2^n binary n-tuples in decreasing order of their occurrence probabilities, then two any complementary n-tuples are always placed at symmetric positions with respect to the middle line. Taking into account that $u + u^c = 2^n - 1$, for all $u \in \{0,1\}^n$, this is equivalent to saying that the sum of any pair of n-tuples placed at symmetric positions with respect to the middle line, is always $2^n - 1$. Figure 3 illustrates this fact for $n = 3$.

Lemma 2 *Let $n \geq 1$ and $1 \leq k \leq 2^n$. Let $u \in \{0,1\}^n$. Then u is k-top iff u^c is k-bottom; u is k-bottom iff u^c is k-top; u is k-jumping iff u^c is k-jumping.*

Proof. Using Lemma 1 the proof is straightforward. □

The following two theorems state some elementary properties of the k-top and k-bottom binary n-tuples.

Theorem 3 *Let $n \geq 1$ and $1 \leq k \leq 2^n$. Let $u \in \{0,1\}^n$. Then*
(i) $\mathcal{T}^1(n) = \{0\}$ and $\mathcal{T}^{2^n}(n) = \{0,1\}^n$.
(ii) $\mathcal{T}^1(n) \subseteq \mathcal{T}^2(n) \subseteq \cdots \subseteq \mathcal{T}^{2^n}(n)$.
(iii) For all k such that $1 \leq k \leq 2^{n-1}$, if $u \in \mathcal{T}^k(n)$ then $u_1 = 0$ and the number of 0-bits in u is greater than or equal to the number of 1-bits in u.

Proof. (i) Using Definition 3-(i) and taking into account that 0 is the maximum element in the poset I_n (see Example 5) the proof is straightforward. (ii) These set inclusions immediately follow from Definition 3-(i). (iii) Using (ii) we get that for all k such that $1 \leq k \leq 2^{n-1}$: $\mathcal{T}^k(n) \subseteq \mathcal{T}^{2^{n-1}}(n) = \mathcal{T}(n)$. Finally, using the positional characterization for top n-tuples stated by Theorem 2-(i), the proof is concluded. □

Theorem 4 *Let $n \geq 1$ and $1 \leq k \leq 2^n$. Let $u \in \{0,1\}^n$. Then*
(i) $\mathcal{B}^1(n) = \{2^n - 1\}$ and $\mathcal{B}^{2^n}(n) = \{0,1\}^n$.
(ii) $\mathcal{B}^1(n) \subseteq \mathcal{B}^2(n) \subseteq \cdots \subseteq \mathcal{B}^{2^n}(n)$.
(iii) For all k such that $1 \leq k \leq 2^{n-1}$, if $u \in \mathcal{B}^k(n)$ then $u_1 = 1$ and the number of 1-bits in u is greater than or equal to the number of 0-bits in u.

Proof. Using Lemma 2 and Theorem 3 the proof is straightforward. □

Now, we give a necessary and sufficient condition for the k-top binary n-tuples. First, note that the k-top condition, i.e., u is k-top if and only if its occurrence probability is always among the k largest ones (Definition 3-(i)), is equivalent to saying that the number of binary n-tuples whose occurrence probabilities are always less than or equal to $\Pr\{u\}$ is at least $2^n - k$. Hence, the proposed question is reduced to determine the number of binary strings v intrinsically less than or equal to u. From now on, we denote the set of these binary n-tuples as follows

$$C^u = \{v \in \{0,1\}^n \mid u \succeq v\} = \{v \in \{0,1\}^n \mid \Pr\{u\} \geq \Pr\{v\}, \; \forall \{p_i\}_{i=1}^n \; s.t. \; (3)\}$$

and we denote the cardinality of this set, as usual, by $|C^u|$.

The set C^u has been characterized from different algorithms that generate all the binary strings intrinsically less than or equal to u. For instance, in [8] we can find such an algorithm which uses the decimal representation of the binary strings for generating all the elements of C^u as set union of certain half-closed intervals of integer numbers. Alternatively, in the next theorem we use the positions of the 1-bits of the binary strings, for determining the number $|C^u|$ of elements of C^u. First, we set the following notation.

Definition 5 *Let $n \geq 1$ and $u = (u_1, \dots, u_n) \in \{0,1\}^n$. Then the vector of positions of 1s of u is the vector of positions of its 1-bits, displayed in increasing order from the left-most position to the right-most position. For all binary n-tuple u with Hamming weight $m \geq 1$, we denote*

$$u = [i_1, \dots, i_m]_n, \quad 1 \leq i_1 < \cdots < i_m \leq n$$

if and only if

$$u_i = 1, \;\; \textit{for all} \;\; i \in \{i_1, \dots, i_m\}; \qquad u_i = 0, \;\; \textit{otherwise}.$$

Example 8. For $n = 6$ and $u = (0,1,1,0,1,1)$ we have

$$m = w_H(u) = 4, \quad u = [i_1, i_2, i_3, i_4]_6 = [2,3,5,6]_6 .$$

Theorem 5 *Let $n \geq 1$ and $u \in \{0,1\}^n$. Let $w_H(u) = m \geq 1$ and let $u = [i_1, \dots, i_m]_n$ be the vector of positions of 1s in u. Then the number of binary n-tuples whose occurrence probabilities are less than or equal to $\Pr\{u\}$ (for all set $\{p_i\}_{i=1}^n$ of parameters satisfying (3)) is given by*

$$|C^u| = 2^n \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{-j_m}.$$

Proof. First, using Corollary 1 we have that if $u \succeq v$ then the Hamming weight of v must be greater than or equal to the Hamming weight of u. That is,

$$v \in C^u \quad \Rightarrow \quad w_H(v) = t \geq m = w_H(u).$$

The matrix description IOC (Theorem 1) of the intrinsic order (expressed in terms of the binary representations of u, v) can be reformulated (in terms of the vectors of positions of the 1-bits of u, v) as follows:

$$u = [i_1, \ldots, i_m]_n \succeq [j_1, \ldots, j_t]_n = v, \quad (w_H(v) = t \geq m = w_H(u))$$

if, and only if, v contains at least one 1-bit among the positions 1 and i_1, at least two 1-bits among the positions 1 and i_2,..., at least $m-1$ 1-bits among the positions 1 and i_{m-1}, and at least m 1-bits among the positions 1 and i_m.

In particular, imposing the additional restriction that the Hamming weight of v equals the Hamming weight of u (i.e., $w_H(v)$ takes the minimum possible value m), then we get

$$u = [i_1, \ldots, i_m]_n \succeq [j_1, \ldots, j_m]_n = v, \quad (w_H(v) = m = w_H(u))$$

if, and only if, v contains at least one 1-bit among the positions 1 and i_1, at least two 1-bits among the positions 1 and i_2,..., at least $m-1$ 1-bits among the positions 1 and i_{m-1}, exactly m 1-bits among the positions 1 and i_m, and (if $i_m < n$) v has no 1-bits among the positions $i_m + 1$ and n. In other words,

$$u = [i_1, \ldots, i_m]_n \succeq [j_1, \ldots, j_m]_n = v, \quad (w_H(v) = m = w_H(u)), \quad \text{iff}$$

$$1 \leq j_1 \leq i_1, \; j_1 + 1 \leq j_2 \leq i_2, \ldots, j_{m-1} + 1 \leq j_m \leq i_m. \tag{5}$$

Note that Equation (5) provides us with a simple algorithm for generating all binary n-tuples v such that v is intrinsically less than or equal to u and it has the same Hamming weight as u. Therefore, the number of those n-tuples is given by

$$|\{v \in C^u \mid w_H(v) = w_H(u)\}| = \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 1. \tag{6}$$

On one hand, note that all the n-tuples v generated by Equation (5) (i.e., those belonging to the subset of C^u in Equation (6)), such that $j_m < n$, obviously satisfy $v_{j_m+1} = \cdots = v_n = 0$. On the other hand, note that the substitution of 0s by 1s in any n-tuple v such that $u \succeq v$ does not avoid the IOC condition, because this substitution change the $\binom{0}{0}$ and $\binom{1}{0}$ columns of matrix M_v^u into $\binom{0}{1}$ and $\binom{1}{1}$ columns, respectively. Hence, to obtain all the binary strings of the set C^u, it is enough to assign, in all possible ways, both values, 0 or 1, to any of the $n - j_m$ null right-most components $v_{j_m+1}, \cdots, v_n$ of all the binary strings v generated by Equation (5). Since there are exactly 2^{n-j_m} different ways of assigning these values, and since by this procedure we generate all elements of C^u without repetitions, then the number of binary n-tuples v

intrinsically less than or equal to u can be immediately obtained from Equation (6) as follows

$$|C^u| = \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{n-j_m} = 2^n \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{-j_m},$$

as was to be shown. □

Now, we can present the characterization of the k-top binary n-tuples.

Theorem 6 *Let $n \geq 1$, $1 \leq k \leq 2^n$ and $u \in \{0,1\}^n$. Let $w_H(u) = m \geq 1$ and let $u = [i_1, \ldots, i_m]_n$ be the vector of positions of 1s in u. Then*

$$u \in \mathcal{T}^k(n) \;\Leftrightarrow\; 2^n \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{-j_m} \geq 2^n - k.$$

Proof. The binary string u is k-top if, and only if, its occurrence probability $\Pr\{u\}$ is always among the k largest ones (Definition 3-(i)) if, and only if, the number of binary n-tuples v whose occurrence probabilities $\Pr\{v\}$ are always less than or equal to $\Pr\{u\}$ is greater than or equal to $2^n - k$ if, and only if, the number of binary strings v intrinsically less than or equal to u is greater than or equal to $2^n - k$ if, and only if, the cardinality of the set C^u is greater than or equal to $2^n - k$. Thus, using Theorem 5, we get

$$u \in \mathcal{T}^k(n) \;\Leftrightarrow\; |C^u| = 2^n \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{-j_m} \geq 2^n - k,$$

as was to be shown. □

Each one of the two following examples illustrates Theorems 5 & 6.

Example 9. For $n = 3$, $k = 6$ and $u = (1,0,0) \equiv 4$, we get (in accordance with Example 7 and Figure 3)

$$m = w_H(u) = 1, \quad u = [i_1]_3 = [1]_3,$$

and using Theorem 5, we have

$$|C^u| = 2^n \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{-j_m} = 2^3 \sum_{j_1=1}^{1} 2^{-j_1} = 2^3 \cdot 2^{-1} = 4,$$

indeed, we have

$$C^u = \{4,5,6,7\},$$

and, finally, using Theorem 5, we get

$$u \in \mathcal{T}^6(3) \;\Leftrightarrow\; |C^u| = 4 \geq 2^3 - 6 = 2.$$

Example 10. For $n = 5$ and $u = (0, 1, 0, 1, 1) \equiv 11$, we get

$$m = w_H(u) = 3, \quad u = [i_1, i_2, i_3]_5 = [2, 4, 5]_5\,,$$

and using Theorem 5, we have

$$\begin{aligned} |C^u| &= 2^n \sum_{j_1=1}^{i_1} \sum_{j_2=j_1+1}^{i_2} \cdots \sum_{j_m=j_{m-1}+1}^{i_m} 2^{-j_m} = 2^5 \sum_{j_1=1}^{2} \sum_{j_2=j_1+1}^{4} \sum_{j_3=j_2+1}^{5} 2^{-j_3} \\ &= 2^5 \left(2^{-3} + 3 \cdot 2^{-4} + 5 \cdot 2^5\right) = 2^2 + 3 \cdot 2^1 + 5 \cdot 2^0 = 15, \end{aligned}$$

indeed, we have

$$C^u = \{11, 13, 14, 15, 19, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31\}\,,$$

and, finally, using Theorem 5, we get

$$u \in \mathcal{T}^k(5) \;\Leftrightarrow\; |C^u| = 15 \geq 2^5 - k \;\Leftrightarrow\; 17 \leq k \leq 32.$$

Remark 6. The assumption $w_H(u) \geq 1$ in Theorems 5 & 6 excludes the zero n-tuple $u = \left(0, \overset{n}{\ldots}, 0\right)$ (the only n-tuple with weight 0). Anyway, for this special case, using Example 5 or Theorem 3-(i), we get

$$C^{\left(0,\overset{n}{\ldots},0\right)} = \{0,1\}^n\,, \quad \left|C^{\left(0,\overset{n}{\ldots},0\right)}\right| = 2^n \text{ and } \left(0, \overset{n}{\ldots}, 0\right) \in \mathcal{T}^k(n)\,,\ \forall\, 1 \leq k \leq 2^n.$$

4 Conclusion

We have presented new approaches to the analysis of CSBSs, i.e., those systems depending on an arbitrary number n of random Boolean variables. The intrinsic ordering between pairs of binary n-tuples associated to a given CSBS enables one to compare their occurrence probabilities without computing them, just looking at the relative positions of their 0s and 1s. In this context, we have considered a no-disjoint decomposition of the set $\{0,1\}^n$ of binary n-tuples into three kinds: the k-top, k-bottom and k-jumping binary n-tuples. A binary n-tuple is called k-top (k-bottom, respectively) if its occurrence probability is always among the k largest (smallest, respectively) ones ($1 \leq k \leq 2^n$). A binary n-tuple is called k-jumping if it is neither k-top nor k-bottom. These thee kinds of binary strings generalize the top, bottom and jumping binary n-tuples, for which $k = 2^{n-1}$. We have presented some elementary properties of both the k-top and k-bottom binary strings. Next, we have characterized by a simple inequality the k-top binary n-tuples. This inequality exclusively depends on n, k and on the Hamming weight m and the positions of the m 1-bits in the binary n-tuples. In this way, Theorem 6 has completely characterized the k-top binary n-tuples. For characterizing the k-bottom binary n-tuples it is enough to use Lemma 2: u is k-bottom if, and only if, u^c is k-top. For characterizing the k-jumping binary

n-tuples it is enough to use Definition 3-(iii): u is k-jumping if it is neither k-top nor k-bottom. These theoretical results may be applied to the reliability analysis of CSBSs (in particular, technical systems) arising from many scientific or engineering areas. For future works, we can establish new properties and necessary conditions of the k-top and k-bottom binary strings and a strong connection between them and the lexicographic order in $\{0,1\}^n$.

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