Abstract

This paper presents a new procedure to improve the quality of triangular meshes defined on surfaces. The improvement is obtained by an iterative process in which each node of the mesh is moved to a new position that minimizes certain objective function. This objective function is derived from an algebraic quality measures of the local mesh (the set of triangles connected to the adjustable or *free node*). The optimization is done in the *parametric mesh*, where the presence of barriers in the objective function maintains the free node inside the *feasible region*. In this way, the original problem on the surface is transformed into a two-dimensional one on the *parametric space*. In our case, the parametric space is a plane, chosen in terms of the local mesh, in such a way that this mesh can be optimally projected performing a *valid* mesh, that is, without *inverted* elements. Several examples and applications presented in this work show how this technique is capable to improve the quality of triangular surface meshes.

Keywords: mesh generation, mesh smoothing, surface meshes, surface mesh optimization, adaptive meshes.

1 Introduction

For 2-D or 3-D meshes the quality improvement [1] can be obtained by an iterative process in which each node of the mesh is moved to a new position that minimizes an objective function [2]. This function is derived from a quality measure of the local mesh. We have chosen, as a starting point in section 2, a 2-D objective function that presents a barrier in the boundary of the *feasible region* (set of points where the free node could be placed to get a *valid* local mesh, that is, without *inverted elements*). This barrier has an important role because it avoids the optimization algorithm to create a tangled mesh when it starts with a valid one. Nevertheless, objective functions constructed by algebraic quality measures are only directly applicable to inner nodes of 2-D or 3-D meshes, but not to its boundary nodes. To overcome this problem, the local mesh, $M(p)$, sited on a surface Σ, is orthogonally projected on a plane P (the existence and search of this plane will be discuss in section 3) in such a way that it performs a valid local mesh $N(q)$. Therefore, it can be said that $M(p)$ is *geometrically conforming* with respect to P [3]. Here p is the free node on Σ and q is its projection on P. The optimization of $M(p)$ is got by the appropriated optimization of $N(q)$. To do this we try to get *ideal* triangles in $N(q)$ that become equilateral in $M(p)$. In general, when the local mesh $M(p)$ is on a surface, each triangle is placed on a different plane and it is not possible to define a feasible region on Σ. Nevertheless, this region is perfectly defined in $N(q)$ as it is analyzed in section 2.1.

To construct the objective function in $N(q)$, it is first necessary to define the objective function in $M(p)$ and, afterward, to establish the connection between them. A crucial aspect for this construction is to keep the barrier of the 2-D objective function. This is done with a suitable approximation in the process that transforms the original problem on Σ into an entirely two-dimensional one on P. We develop this approximation in section 2.2.

The optimization of $N(q)$ becomes a two-dimensional iterative process. The optimal solutions of each two-dimensional problem form a sequence $\{\mathbf{x}^k\}$ of points belonging to P. We have checked in many numerical test that $\{\mathbf{x}^k\}$ is always a convergent sequence. It is important to underline that this iterative process only takes into account the position of the free node in a discrete set of points, the points on Σ corresponding to $\{\mathbf{x}^k\}$ and, therefore, it is not necessary that the surface is smooth. Indeed, the surface determined by the piecewise linear interpolation of the initial mesh is used as a reference to define the geometry of the domain.

If the node movement only responds to an improvement of the quality of the mesh, it can happen that the optimized mesh loses details of the original surface. To avoid this problem, every time the free node p is moved on Σ, the optimization process only allows a small distance between the centroid of the triangles of $M(p)$ and the underlaying surface (the true surface, if it is known, or the piece-wise linear interpolation, if it is not).

There are several alternatives to the previous method. For example, Garimella et al. [4] develop a method to optimize meshes in which the nodes of the optimized mesh are kept close to the original positions by imposing the Jacobians of the current and original meshes to be also close. Frey et al. [5] get a control of the gap between the mesh and the surface by modifying the element-size (subdividing the longest edges and collapsing the shortest ones) in terms of an approximation of the smallest principal curvatures radius associated to the nodes. Rassineux et al. [6] also use the smallest principal curvatures radius to estimate the element-size compatible with a prescribed gap error. They construct a geometrical model by using the Hermite diffuse interpolation in which local operations like edge swapping, node removing, edge splitting, etc. are made to adapt the mesh size and shape. More accurate approaches, that have into account the directional behavior of the surface, have been considered in by Vigo [7] and, recently, by Frey in [8].

Application of our proposed optimization technique is shown in section 4.

2 Construction of the objective function

As it is shown in [2], [9], and [10] we can derive optimization functions from *algebraic quality measures* of the elements belonging to a local mesh. Let us consider a triangular mesh defined in $\mathbb{R}^2$ and let t be an triangle in the physical space whose vertices are given by $\mathbf{x}_k = (x_k, y_k)^T \in \mathbb{R}^2$, $k = 0, 1, 2$. First, we are going to introduce an algebraic quality measure for t. Let t_R be the reference triangle with vertices $\mathbf{u}_0 = (0,0)^T$, $\mathbf{u}_1 = (1,0)^T$, and $\mathbf{u}_2 = (0,1)^T$. If we choose $\mathbf{x}_0$ as the translation vector, the affine map that takes t_R to t is $\mathbf{x} = A\mathbf{u} + \mathbf{x}_0$, where A is the Jacobian matrix of the affine map referenced to node $\mathbf{x}_0$, given by $A = (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0)$. We will denote this type of affine maps as $t_R \xrightarrow{A} t$. Let now t_I be an *ideal* triangle (not necessarily equilateral) whose vertices are $\mathbf{w}_k \in \mathbb{R}^2$, $(k = 0, 1, 2)$ and let $W_I = (\mathbf{w}_1 - \mathbf{w}_0, \mathbf{w}_2 - \mathbf{w}_0)$ be the Jacobian matrix, referenced to node $\mathbf{w}_0$, of the affine map $t_R \xrightarrow{W_I} t_I$; then, we define $S = AW_I^{-1}$ as the weighted Jacobian matrix of the affine map $t_I \xrightarrow{S} t$. In the particular case that t_I was the equilateral triangle t_E, the Jacobian matrix $W_I = W_E$ will be defined by $\mathbf{w}_0 = (0,0)^T$, $\mathbf{w}_1 = (1,0)^T$ and $\mathbf{w}_2 = (1/2, \sqrt{3}/2)^T$.

We can use matrix norms, determinant or trace of S to construct algebraic quality measures of t. For example, the Frobenius norm of S, defined by $|S| = \sqrt{tr\,(S^T S)}$, is specially indicated because it is easily computable. Thus, it is shown in [1] that $q_\eta = \frac{2\sigma}{|S|^2}$ is an algebraic quality measure of t , where $\sigma = \det(S)$. We use this quality measure to construct an objective function. Let $\mathbf{x} = (x, y)^T$ be the position vector of the free node, and let S_m be the weighted Jacobian matrix of the m-th triangle of a valid local mesh of M triangles. The objective function associated to m-th triangle is $\eta_m = \frac{|S_m|^2}{2\sigma_m}$, and the corresponding objective function for the local mesh is the n-norm of $(\eta_1, \eta_2, \ldots, \eta_M)$,

$$\left|K_\eta\right|_n(\mathbf{x}) = \left[\sum_{m=1}^{M} \eta_m^n(\mathbf{x})\right]^{\frac{1}{n}} \tag{1}$$

This objective function presents a barrier in the boundary of the feasible region that avoids the optimization algorithm to create a tangled mesh when it starts with a valid one.

Previous considerations and definitions are only directly applicable for 2-D (or 3-D) meshes, but some of them must be properly adapted when the meshes are located on an arbitrary surface. For example, the concept of valid mesh is not clear in this situation because neither the concept of inverted element is. We will deal with these questions in next subsections.

2.1 Similarity Transformation for Surface and Parametric Meshes

Suppose that for each local mesh $M(p)$ placed on the surface Σ, that is, with all its nodes on Σ, it is possible to find a plane P such that the orthogonal projection of $M(p)$ on P is a valid mesh $N(q)$. Moreover, suppose that we define the axes in such a way that the x, y-plane coincide with P. If, in the feasible region of $N(q)$, it is possible to define the surface Σ by the parametrization $\mathbf{s}(x, y) = (x, y, f(x, y))$, where f is a continuous function, then, we can optimize $M(p)$ by an appropriate optimization of $N(q)$. We will refer to $N(q)$ as the *parametric mesh*. The basic idea consists on finding the position $\bar{q}$ in the feasible region of $N(q)$ that makes $M(p)$ be an optimum local mesh. To do this, we search *ideal* elements in $N(q)$ that become equilateral in $M(p)$. Let $\tau \in M(p)$ be a triangular element on Σ whose vertices are given by $\mathbf{y}_k = (x_k, y_k, z_k)^T$, ($k = 0, 1, 2$) and t_R be the reference triangle in P (see Figure 1). If we choose $\mathbf{y}_0$ as the translation vector, the affine map $t_R \xrightarrow{A_\pi} \tau$ is $\mathbf{y} = A_\pi \mathbf{u} + \mathbf{y}_0$, where A_π is its Jacobian matrix, given by

$$A_\pi = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \\ z_1 - z_0 & z_2 - z_0 \end{pmatrix} \tag{2}$$

Now, consider that $t \in N(q)$ is the orthogonal projection of τ on P. Then, the vertices of t are $\mathbf{x}_k = \Pi \mathbf{y}_k = (x_k, y_k)^T$, ($k = 0, 1, 2$), where $\Pi = (\mathbf{e}_1, \mathbf{e}_2)^T$ is 2×3 matrix of the affine map $\tau \xrightarrow{\Pi} t$, being $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the canonical basis in $\mathbb{R}^3$ (the associated projector from $\mathbb{R}^3$ to P, considered as a subspace of $\mathbb{R}^3$, is $\Pi^T \Pi$). Taking $\mathbf{x}_0$ as translation vector, the affine map $t_R \xrightarrow{A_P} t$ is $\mathbf{x} = A_P \mathbf{u} + \mathbf{x}_0$, where $A_P = \Pi A_\pi$ is its Jacobian matrix

$$A_P = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} \tag{3}$$

Therefore, the 3×2 matrix of the affine map $t \xrightarrow{T} \tau$ is

$$T = A_\pi A_P^{-1} \tag{4}$$

Let V_π be the subspace spanned by the column vectors of A_π and let π be the plane defined by V_π and the point $\mathbf{y}_0$. Our goal is to find the *ideal* triangle $t_I \subset P$, moving q on P, such that t_I is mapped by T into an equilateral one, $\tau_E \subset \pi$. In general, the strict fulfillment of this requirement is only possible if $N(q)$ is formed by a unique triangle.

Due to rank$(A_\pi) =$ rank$(A_P) = 2$, it exists a unique factorization $A_\pi = QR$, where Q is an orthogonal matrix and R is an upper triangular one with $[R]_{ii} > 0$ ($i = 1, 2$). The columns of the 3×2 matrix Q define an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ that spans V_π, so we can see Q as the matrix of the affine map $t_R \xrightarrow{Q} \tau_R$ and R as the 2×2 Jacobian matrix of the affine map $\tau_R \xrightarrow{R} \tau$ (see Figure 1). As $t_R \xrightarrow{W_E} t_E$ and Q is an orthogonal matrix that keeps the angles and norms of the vectors, then $t_E \xrightarrow{Q} \tau_E$ and, therefore

$$QW_E = A_\pi R^{-1} W_E \tag{5}$$

is the 3×2 Jacobian matrix of affine map $t_R \overset{QW_E}{\longrightarrow} \tau_E$. On the other hand, we define on the plane π

$$S = RW_E^{-1} \tag{6}$$

as the 2×2 weighted Jacobian matrix of the affine map that transforms the equilateral triangle into the physical one, that is, $\tau_E \overset{S}{\rightarrow} \tau$.

We have chosen as ideal triangle in π the equilateral one ($\tau_I = \tau_E$), then, the Jacobian matrix W_I of the affine map $t_R \overset{W_I}{\rightarrow} t_I$ is calculated by imposing the condition $TW_I = QW_E$, because $t_R \overset{TW_I}{\longrightarrow} \tau_I$ and $t_R \overset{QW_E}{\longrightarrow} \tau_E$. Taking into account (5), it yields

$$TW_I = A_\pi R^{-1} W_E \tag{7}$$

and, from (4), we obtain

$$W_I = A_P R^{-1} W_E \tag{8}$$

so we define on P the *ideal-weighted* Jacobian matrix of the affine map $t_I \overset{S_I}{\rightarrow} t$ as $S_I = A_P W_I^{-1}$. From (8) it results

$$S_I = A_P W_E^{-1} R A_P^{-1} \tag{9}$$

and, from (6)

$$S_I = A_P W_E^{-1} S W_E A_P^{-1} = A_P W_E^{-1} S \left(A_P W_E^{-1}\right)^{-1} = S_E S S_E^{-1} \tag{10}$$

where $S_E = A_P W_E^{-1}$ is the *equilateral-weighted* Jacobian matrix of the affine map $t_E \overset{S_E}{\rightarrow} t$. Finally, from (10), we obtain the next similarity transformation.

$$S = S_E^{-1} S_I S_E \tag{11}$$

Therefore, it can be said that the matrices S and S_I are *similar*.

2.2 Optimization on the Parametric Space

It might be used S, as it is defined in (6), to construct the objective function and, then, solve the optimization problem. Nevertheless, this procedure has important disadvantages. First, the optimization of $M(p)$, working on the true surface, would require the imposition of the constraint $p \in \Sigma$. It would complicate the resolution of the problem because, in many cases, Σ is not defined by a smooth function. Moreover, when the local mesh $M(p)$ is on a curved surface, each triangle is sited on a different plane and the objective function, constructed from S, lacks barriers. It is impossible to define a feasible region in the same way as it was done at the beginning of this section. Indeed, all the positions of the free node, except those that make $\det(S) = 0$ for any triangle, produce correct triangulations of $M(p)$. However, for many purposes as, for example, to construct a 3-D mesh from the surface triangulation, there are unacceptable positions of the free node.

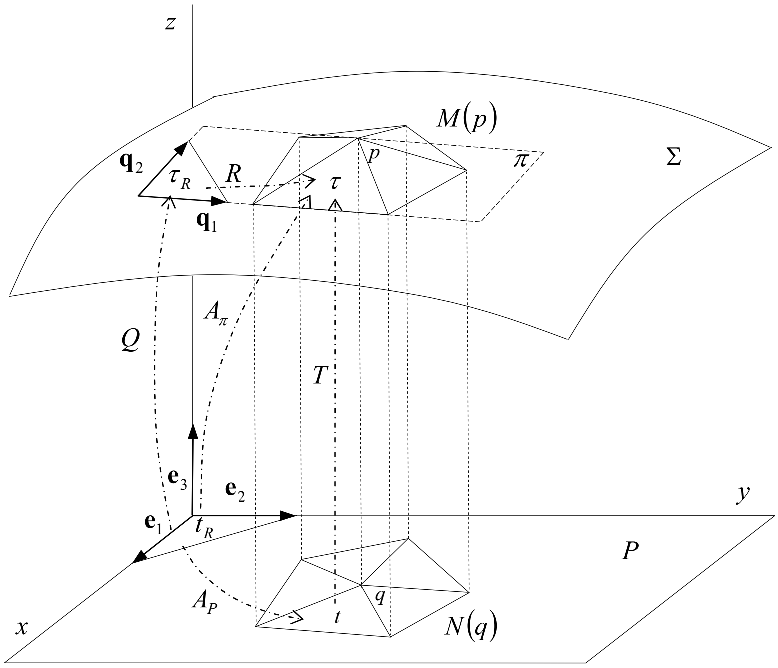

Figure 1: Local surface mesh $M(p)$ and its associated parametric mesh $N(q)$

To overcome these difficulties we propose to carry out the optimization of $M(p)$ in an indirect way, working on $N(q)$. With this approach the movement of the free node will be restricted to the feasible region of $N(q)$, which avoids to construct unacceptable surface triangulations. It all will be carried out using an approximate version of the similarity transformation given in (11).

Let us consider that $\mathbf{x} = (x, y)^T$ is the position vector of the free node q, sited on the plane P. If we suppose that Σ is parametrized by $\mathbf{s}(x, y) = (x, y, f(x, y))$, then, the position of the free node p on the surface is given by $\mathbf{y} = (x, y, f(x, y))^T = (\mathbf{x}, f(\mathbf{x}))^T$.

Note that $S_E = A_P W_E^{-1}$ only depends on $\mathbf{x}$ because W_E is constant and A_P is a function of $\mathbf{x}$. Besides, $S_I = A_P W_I^{-1}$ depends on $\mathbf{y}$, due to $W_I = A_P R^{-1} W_E$, and R is a function of $\mathbf{y}$. Thus, we have $S_E(\mathbf{x})$ and $S_I(\mathbf{y})$. We shall optimize the local mesh $M(p)$ by an iterative procedure maintaining constant $W_I(\mathbf{y})$ in each step. To do this, at the first step, we fix $W_I(\mathbf{y})$ to its initial value, $W_I^0 = W_I(\mathbf{y}^0)$, where $\mathbf{y}^0$ is given by the initial position of p. So, if we define $S_I^0(\mathbf{x}) = A_P(\mathbf{x})(W_I^0)^{-1}$, we approximate the similarity transformation (11) as

$$S^0(\mathbf{x}) = S_E^{-1}(\mathbf{x})\, S_I^0(\mathbf{x})\, S_E(\mathbf{x}) \tag{12}$$

Now, the construction of the objective function is carried out in a standard way, but using S^0 instead of S. So, we obtain the objective function for a given triangle $\tau \subset \pi$

$$\eta^0(\mathbf{x}) = \frac{|S^0(\mathbf{x})|^2}{2\sigma^0(\mathbf{x})} \tag{13}$$

where $\sigma^0(\mathbf{x}) = \det(S^0(\mathbf{x}))$.

With this approach the optimization of the local mesh $M(p)$ is transformed into a two-dimensional problem without constraints, defined on $N(q)$, and, therefore, it can be solved with low computational cost. Furthermore, if we write W_I^0 as $A_P^0(R^0)^{-1}W_E$, where $A_P^0 = A_P(\mathbf{x}^0)$ and $R^0 = R(\mathbf{y}^0)$, it is straightforward to show that S^0 can be simplified as

$$S^0(\mathbf{x}) = R^0\left(A_P^0\right)^{-1} S_E(\mathbf{x}) \tag{14}$$

and our objective function for the local mesh is

$$\left|K_\eta^0\right|_n(\mathbf{x}) = \left[\sum_{m=1}^{M}\left(\eta_m^0\right)^n(\mathbf{x})\right]^{\frac{1}{n}} \tag{15}$$

Let now analyze the behavior of the objective function when the free node crosses the boundary of the feasible region. If we denote $\alpha_P = \det(A_P)$, $\alpha_P^0 = \det(A_P^0)$, $\rho^0 = \det(R^0)$, $\omega_E = \det(W_E)$ and taking into account (14), we can write $\sigma^0 = \rho^0\left(\alpha_P^0\right)^{-1}\alpha_P\omega_E^{-1}$. Note that ρ^0, α_P^0, and ω_E are constants, so η^0 has a singularity when $\alpha_P = 0$, that is, when q is placed on the boundary of the feasible region of $N(q)$. This singularity determines a barrier in the objective function that prevents the optimization algorithm to take the free node outside this region. This barrier does not appear if we use the exact weighted Jacobian matrix S, given in (6), due to $\det(R) = R_{11}R_{22} > 0$.

Suppose that $\mathbf{x}^1 = \bar{\mathbf{x}}^0$ is the minimizing point of (15). As this objective function has been constructed by keeping $\mathbf{y}$ in its initial position, $\mathbf{y}^0$, then $\mathbf{x}^1$ is only the first approximation to our problem. This result is improved updating the objective function at $\mathbf{y}^1 = (\mathbf{x}^1, f(\mathbf{x}^1))^T$ and, then, computing the new minimizing position, $\mathbf{x}^2 = \bar{\mathbf{x}}^1$. This local optimization process is repeated, obtaining a sequence $\left\{\mathbf{x}^k\right\}$ of optimal points, until a convergence criteria is verified. We have experimentally verified in numerous tests, involving continuous functions to define the surface Σ, that this algorithm converges.

Let us consider P as an optimal projection plane (this aspect will be discussed in next section). In order to prevent a loss of the details of the original geometry, our optimization algorithm evaluates the difference of heights ($[\Delta z]$) between the centroid of the triangles of $M(p)$ and the reference surface, every time a new position $\mathbf{x}^k$ is calculated. If this distance exceeds a threshold, $\Delta(p)$, the movement of the node is aborted and the previous position is stored. This threshold $\Delta(p)$ is established attending to the size of the elements of $M(p)$. In concrete, the algorithm evaluates the average distance between the free node and the nodes connected to it, and takes $\Delta(p)$ as percentage of this distance.

3 Search of the optimal projection plane

The former procedure needs a plane in which the local mesh, $M(p)$, is projected conforming a valid mesh, $N(q)$. If this plane exists it is not unique, because a small rotation of the coordinate system produces another valid projection plane, that is, another plane in which $N(q)$ is valid. We have observed that the number of iterations required by our procedure depends on the chosen plane. In general, this number is less if the plane is well *faced* to $M(p)$. We have to find the rotation of reference system x, y, z such that the new x', y'-plane, P', is optimal with respect to a suitable criterion.

We will denote $N(q')$ as the projection of $M(p)$ onto P' and t' the projection of the physical triangle $\tau \in M(p)$ onto P'. Let $A'_P = (\mathbf{x}'_1 - \mathbf{x}'_0, \mathbf{x}'_2 - \mathbf{x}'_0)$ be the matrix associated to the affine map that takes the reference element defined on P' to t', then, the area of t' is given by $\frac{1}{2}|\alpha'_P|$ where $\alpha'_P = \det(A'_P)$.

Our goal is to find a coordinate system rotation such that $\sum_{m=1}^{M} \alpha'_{P_m}$ is maximum satisfying the constraints $\alpha'_{P_m} = \det\left(A'_{P_m}\right) > 0$ for all the triangles of $N(q')$, that is, $m = 1, ..., M$. In [11] a method to determine a projection plane is considered but without the enforcement of these constraints.

According to Euler's rotation theorem, any rotation may be described using three angles. The so-called x-*convention* is the most common definition. In this convention, the rotation is given by Euler angles (ϕ, θ, ψ), where the first rotation is by an angle $\phi \in [0, 2\pi]$ about the z-axis, the second is by an angle $\theta \in [0, \pi]$ about the x-axis, and the third is by an angle $\psi \in [0, 2\pi]$ about the z-axis (again).

Let $\Phi(\phi, \theta, \psi)$ be the Euler's rotation matrix such that $\mathbf{y}' = \Phi\mathbf{y}$, then, the Jacobian matrix $A_\pi = (\mathbf{y}_1 - \mathbf{y}_0, \mathbf{y}_2 - \mathbf{y}_0)$ associated to the triangle τ of $M(p)$, defined in (2), can be spanned on the rotated coordinate system as $A'_\pi = (\mathbf{y}'_1 - \mathbf{y}'_0, \mathbf{y}'_2 - \mathbf{y}'_0) = \Phi A_\pi$. Thus, the Jacobian matrix A'_P is written as $A'_P = \Pi A'_\pi = \Pi\Phi A_\pi$. With these considerations it is easy to proof that the value of α'_P is

$$\alpha'_P = \det(\Pi\Phi A_\pi) = m_1 \sin(\phi)\sin(\theta) + m_2 \sin(\theta)\cos(\phi) + m_3 \cos(\theta) \qquad (16)$$

where m_i is the minor obtained by deleting the i-th row of A_π. Note that equation (16) only depends on ϕ and θ angles, as was to be expected.

Although the above maximization problem can be solved taken into account the constraints, we propose an unconstrained approach.

Let us consider, as a first attempt, the objective function $\sum_{m=1}^{M} (\alpha'_{P_m})^{-1}(\phi, \theta)$. The minimization of this function tends to maximize the values of α'_{P_m} and, due to the barrier that appears when $\alpha'_{P_m} = 0$ for some triangle of $N(q')$, the values of α'_{P_m} are maintained positive if the minimization algorithm starts at an interior point, that is, a point (ϕ_0, θ_0) belonging to the set Ψ of angles (ϕ, θ) such that $\alpha'_{P_m}(\phi, \theta) > 0$ for $(m = 1, ..., M)$. On the other hand, if any $\alpha'_{P_m} < 0$ the barrier prevents to reach the required minimum. In next paragraph we propose a method to find an interior point (ϕ_0, θ_0) of Ψ to be used as a starting point in the minimization algorithm.

Let $G = [\mathbf{g}_m]$ be the $3 \times M$ matrix formed by the vectors, $\mathbf{g}_m$, normal to the triangles of $M(p)$. A solution of the inequality system (if it exists) $G^T\mathbf{g} > \mathbf{0}$ provides a direction, defined by vector $\mathbf{g}$, such that all the triangles of $M(p)$ can be projected on a plane, normal to the unitary vector $\mathbf{n} = \frac{\mathbf{g}}{\|\mathbf{g}\|}$, so that $\alpha'_{P_m} > 0$ for $(m = 1, ..., M)$.

Then, it only remains to find the angles ϕ_0 and θ_0 in which the coordinate system needs to be rotated to get the z' axis to point in the direction of $\mathbf{n}$. More precisely, the angles ϕ_0 and θ_0 are the solution of the equation $\Phi^T(\phi_0, \theta_0, 0)\,\mathbf{e}_3 = \mathbf{n}$, where $\mathbf{e}_3 = (0, 0, 1)^T$. If the inequality system has not solution, then, there is not any valid projection plane for this local mesh, against the premise done in section 2.1. In this case, the local optimization procedure maintains the free node p at its initial position.

We have observed that the previous objective function has computational difficulties as the optimization algorithms use discrete steps to search the optimal point. A step leading outside the region Ψ may indicate a decrease in the value of the objective function and take to a false solution. To overcome this problem we propose a modification of the objective function in such a way that it will be regular all over $\mathbb{R}^3$ and its barrier will be "smoothed". The modification consists of substituting α'_{P_m} by $h(\alpha_{P_m})$, where $h(\alpha)$ is the positive and increasing function given by

$$h(\alpha) = \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4\delta^2}) \tag{17}$$

being the parameter $\delta = h(0)$. The behavior of $h(\alpha)$ in function of δ parameter is such that, $\lim\limits_{\delta \to 0} h(\alpha) = \alpha$, $\forall \alpha \geq 0$ and $\lim\limits_{\delta \to 0} h(\alpha) = 0$, $\forall \alpha \leq 0$. The characteristics of h function and its application in the context of mesh untangling and smoothing have been studied in [12], [13]. Thus, the proposed objective function for searching the projection plane is

$$\Omega(\phi, \theta) = \sum_{m=1}^{M} \frac{1}{h(\alpha'_{P_m}(\phi, \theta))} \tag{18}$$

A crucial property is that the angles that minimize the original and modified objective functions are nearly identical when δ is *small*. Details about the determination of δ value for 3-D triangulations can be found in [13].

4 Applications

In this section, the proposed technique is applied to smooth the mesh of a scanned object. In particular, we have applied the optimization technique to a mesh of a rocker arm obtained from *http://www.cyberware.com/*. This mesh (see Figure 2) has 80354 triangles and 40177 nodes. The value of the average quality is 0.841 (measured with the quality metric based on the condition number [2]). The optimized mesh is shown in Figure 4. The projection plane is chosen in terms of the local mesh to be analyzed and the norm chosen for the objective function (1) in this application have been $n = 2$. Note the poor quality of the original mesh in several parts of the device. The algorithm increases the mean quality to 0.923 in four iterations, but only needs one iteration to reach a mean quality of 0.907. Another significant data is that average quality of the worst 1000 triangles. It increases from 0.247 to 0.697. Details of the original and optimized meshes are shown in Figure 3 and Figure 4.

Optimized mesh of the rocker arm after four iterations of our procedure

Figure 2: Original mesh of the rocker arm

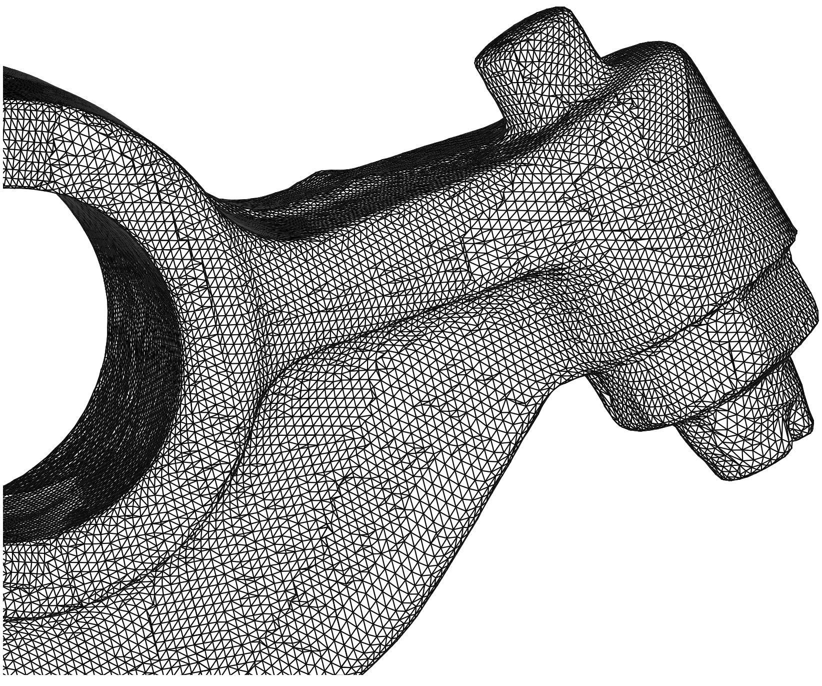

Figure 3: Detail of the original mesh

In Figure 5 it is shown the quality curves for the initial and optimized meshes. These curves are obtained by sorting the elements in increasing order of its quality. The number of events such that the threshold, $\Delta(p)$ is been exceeded, taking $\Delta(p)$ as 10% of average distance between the free node and the nodes connected to it, has been 259 in the first iteration, 379 in the second one, 395 in the third one and 408 in the fourth one.

5 Conclusions and future research

We have developed an algebraic method to optimize triangulations defined on surfaces. Its main characteristic is that the original problem is transformed into a fully two-dimensional sequence of approximate problems on the parametric space. This characteristic allows the optimization algorithm to deals with surfaces that only need to be continuous. Moreover, the barrier exhibited by the objective function in the parametric space prevents the algorithm to construct unacceptable meshes.

We have also introduced a procedure to find an optimal projection plane (our parametric space) based on the minimization of a suitable objective function. We have observed that correct choice of this plane plays a relevant role.

The optimization process includes a control on the gap between the optimized mesh and the reference surface that avoids to lose details of the original geometry. In this work we have used a piecewise linear interpolation to define the reference surface

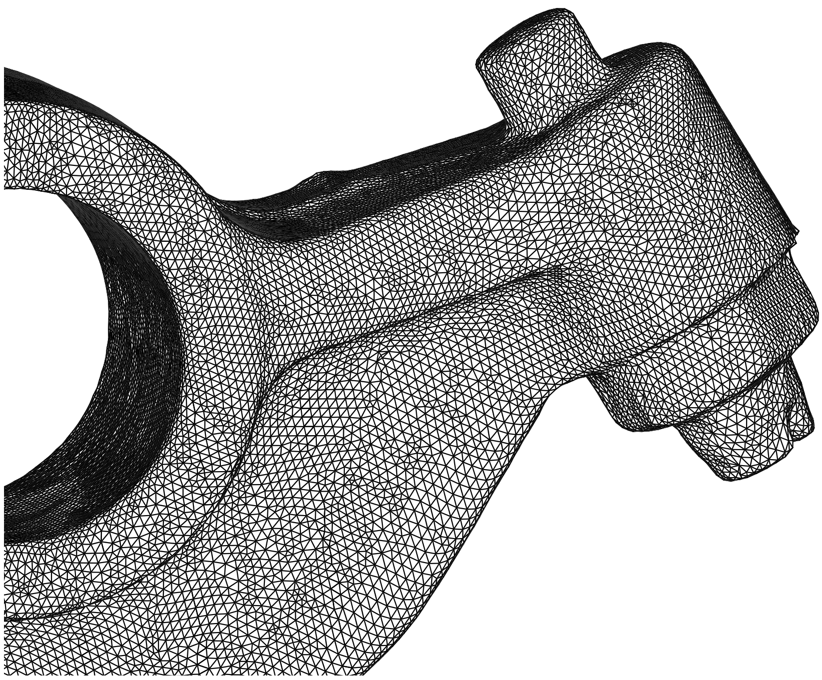

Figure 4: Detail of the optimized mesh

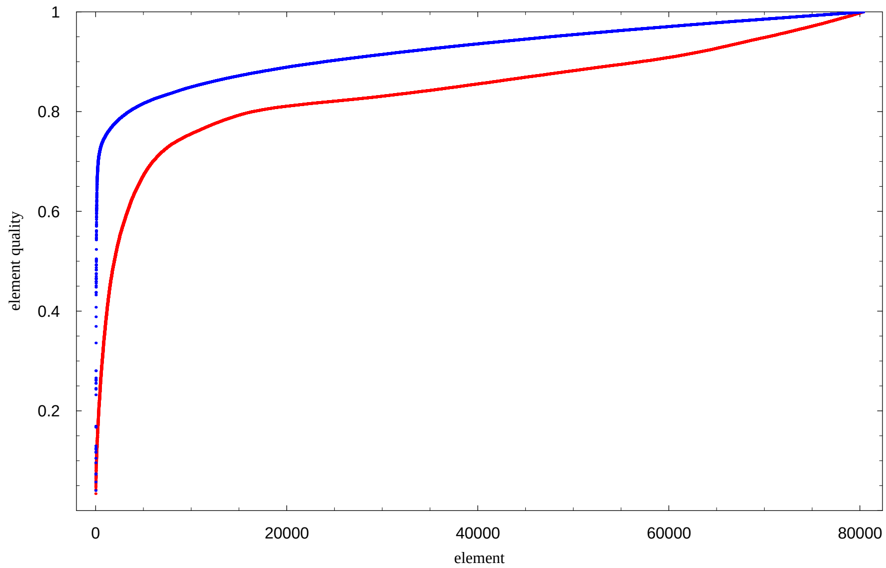

Figure 5: Quality curves for the initial and optimized meshes

when the true surface is not known, but it would be also possible to use a more regular interpolation, for example, the proposed in [6]. Likewise, it would be possible to introduce a more sophisticated stopping criterion for the gap control that takes into account the curvature of the surface [5], [6], [7], [8].

In the present work we have only considered a sole objective function obtained from an isotropic and area independent algebraic quality metric. Nevertheless, the framework that establishes the *algebraic quality measures* [1] provides us the possibility to construct anisotropic and area sensitive objective functions by using a suitable metric.

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