

Updating incomplete factorization preconditioners for shifted linear systems arising in a wind model

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Abstract

The efficiency of a finite element mass-consistent model for wind field adjustment depends on the stability parameter α which allows from a strictly horizontal wind adjustment to a pure vertical one. Each simulation with the wind model leads to the resolution of a linear system of equations, the matrix of which depends on a function $\varepsilon(\alpha)$, i.e., $(M + \varepsilon N) x_\varepsilon = b_\varepsilon$, where M and N are constant, symmetric and positive definite matrices with the same sparsity pattern for a given level of discretization. The estimation of this parameter may be carried out by using genetic algorithms. This procedure requires the evaluation of a fitness function for each individual of the population defined in the searching space of α , that is, the resolution of one linear system of equations for each value of α . Preconditioned Conjugate Gradient algorithm (PCG) is usually applied for the resolution of this type of linear systems due to its good convergence results. In order to solve this set of linear systems, we could either construct a different preconditioner for each of them or use a single preconditioner constructed from the first value of ε to solve all the systems. In this paper, an intermediate approach is proposed. An incomplete Cholesky factorization of matrix A_ε is constructed for the first linear system and it is updated for each ε at a low computational cost. Numerical experiments related to realistic wind field are presented in order to show the performance of the proposed preconditioning strategy.

Key words: Incomplete factorization, Shifted linear systems, Preconditioning, Conjugate gradient, Wind modelling, Genetic algorithms

1 INTRODUCTION

The application of discretization techniques to problems defined by partial differential equations that model physical phenomena leads to linear systems of equations of which the matrix is sometimes given as a function of a parameter. This is specifically true in the numerical simulation of wind fields with mass consistent models [1,2].

In general, these problems are defined over regions with complex terrain, therefore a suitable discretization of the studied zone is necessary. Here, we have used the technique of Montenegro et al. [3] for constructing tetrahedral meshes such that they are adapted to the terrain orography and have a higher density of nodes near the terrain surface. Moreover some regions may need an additional refinement due, for example, to more accurate approximation in those zones. On the other hand, the combination of the model with a Gaussian plume approach makes the refinement along the trajectory of the plume necessary. So, in general, we are going to work with meshes including elements of very different size. This fact affects the conditioning of the linear systems of equations that arises from this type of discretization in this problem, i.e.,

$$A_\varepsilon x_\varepsilon = b_\varepsilon \tag{1}$$

Thus, a suitable preconditioning technique should be applied for an efficient conjugate gradient iteration. We are interested on the preconditioning of (1) in the particular case

$$A_\varepsilon = M + \varepsilon N \tag{2}$$

where M and N , are symmetric and positive definite matrices and remain constant along the process for a given discretization level. These linear systems appear in each step of the model. Two different cases exist where a sequence of linear systems like (1) must be solved. On the one hand, we have applied genetic algorithms to estimate ε for each given set of station measurements [2,4]. Genetic algorithms (GAs) are optimisation tools based on the natural evolution mechanism. They produce successive trials that have an increasing probability of obtaining a global optimum. The main aspects of GAs are the construction of an initial population, the evaluation of each individual of a population with a fitness function, the selection of the parents of the next generation, the crossover of those parents to create the children and the mutation to increase diversity. In practice, the initial population is randomly generated and we use *iteration limit exceeded* as the stopping criterion. The selection phase allocates an intermediate population on the basis of the evaluation of the fitness function. This evaluation, which is in general related to the difference between the observed and computed wind at the stations, leads us to solve a complete wind simulation for each individual of the population. In other words, we have to solve a linear system of equations like (1) for each

value of ε .

On the other hand, for a given sequence of ε values corresponding to a time interval of a simulation episode, a set of linear systems must also be solved.

Two extreme strategies for preconditioning such linear systems may be applied. Indeed, we could either construct a different preconditioner for each of them and improve the convergence of Preconditioned Conjugate Gradient (PCG) [5] with the consequent high computational cost related to the construction of each preconditioner, or use a single preconditioner constructed with a given value of ε for all the systems. In this latter case the convergence will be getting worse as the value of the parameter moves away from the initial value.

In the particular case of shifted linear systems, Meurant [6] and Benzi et al. [7] propose an intermediate solution by using two type of preconditioners, respectively, which are constructed once at the beginning of the process and updated for each ε at a low computational cost. These strategies also lead to an intermediate rate of convergence between the above extreme options. On the one hand, Meurant studied the case of a shifted matrix of the form $A_\varepsilon = M + \varepsilon D$, with D being a diagonal matrix, and updated the preconditioner from an incomplete Cholesky factorization of M . Here, we generalize Meurant's algorithm to the case of a generalised shifted matrix $A_\varepsilon = M + \varepsilon N$, with M and N SPD matrices with the same sparsity pattern arising from a finite element discretization of mass consistent mathematical model. On the other hand, Benzi et al. develop the study of factorised approximate inverses using the SAINV algorithm [8] for the special case of $A_\varepsilon = M + \varepsilon I$, with I being the unit matrix. This approach is oriented to parallel computing, since the construction of the preconditioner may be carried out in parallel. A similar updating method for a generalised shifted matrix using SAINV algorithm is proposed in [9].

The organisation of the paper is as follows. In Section 2, the mass consistent model is presented. It generates a wind velocity field for an incompressible fluid which adjusts to an initial field obtained from experimental measurements and physical considerations. The construction of the initial field may be found in [2] and is not directly involved in the updating of matrix A_ε . Nevertheless, we remark the study of the stability parameter α of the wind model which is directly related to ε and, thus, is the generator of the set of linear systems. The construction of the preconditioner is carried out in terms of an incomplete Cholesky factorization of matrix $A_\varepsilon = M + \varepsilon N$, with the corresponding simplifications that allow it to be update at a reasonable cost. This is described in Section 3. Section 4 is devoted to illustrating the performance of this preconditioner in same numerical experiments. Finally, our conclusions are presented in Section 5.

2 WIND MODEL

This model [1] is based on the continuity equation for an incompressible flow where the air density is constant in the domain Ω and *no-flow-through* conditions on Γ_b (terrain and top) are considered

$$\vec{\nabla} \cdot \vec{u} = 0 \quad \text{in } \Omega \quad (3)$$

$$\vec{n} \cdot \vec{u} = 0 \quad \text{on } \Gamma_b \quad (4)$$

The problem is formulated as a least-square approach in Ω , with $\vec{u}(\tilde{u}, \tilde{v}, \tilde{w})$ to be adjusted

$$E(\vec{u}) = \int_{\Omega} \left[\alpha_1^2 \left((\tilde{u} - u_0)^2 + (\tilde{v} - v_0)^2 \right) + \alpha_2^2 (\tilde{w} - w_0)^2 \right] d\Omega \quad (5)$$

where the interpolated wind $\vec{v}_0 = (u_0, v_0, w_0)$ is obtained from experimental measurements and physical considerations, and α_1, α_2 are the Gauss precision moduli.

In practice, we use the so called stability parameter of the wind model,

$$\alpha = \frac{\alpha_1}{\alpha_2} \quad (6)$$

since the minimum of the functional given by (5) is the same if we divide it by α_2^2 . So, if $\alpha \gg 1$, flow adjustment in the vertical direction predominates. However if $\alpha \ll 1$, flow adjustment occurs primarily in the horizontal plane. Thus, the selection of α allows the air to go over a terrain barrier or around it, respectively. Moreover, the behaviour of mass consistent models in many numerical experiments has shown that they are very sensitive to the value chosen for α . In [4,2] a brief discussion about the selection of α by several authors is presented.

Solving (5) constrained by (3) and (4) is equivalent to find a saddle point (\vec{v}, ϕ) of the Lagrangian

$$E(\vec{v}) = \min_{\vec{u} \in K} \left[E(\vec{u}) + \int_{\Omega} \phi \vec{\nabla} \cdot \vec{u} d\Omega \right] \quad (7)$$

where $\vec{v} = (u, v, w)$, ϕ the Lagrange multiplier and K the set of admissible functions. The Lagrange multipliers technique is used to minimise the problem (7), whose minimum comes to form the Euler-Lagrange equations

$$u = u_0 + \frac{1}{2\alpha_1^2} \frac{\partial \phi}{\partial x}, \quad v = v_0 + \frac{1}{2\alpha_1^2} \frac{\partial \phi}{\partial y}, \quad w = w_0 + \frac{1}{2\alpha_2^2} \frac{\partial \phi}{\partial z} \quad (8)$$

Since α_1 and α_2 are constant in Ω , the variational approach results in an elliptic problem substituting (8) in (3)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \alpha^2 \frac{\partial^2 \phi}{\partial z^2} = -2\alpha_1^2 \left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} \right) \quad \text{in } \Omega \quad (9)$$

We consider Dirichlet condition for *flow-through* boundaries and Neumann condition for terrain and top

$$\phi = 0 \quad \text{on } \Gamma_a \quad (10)$$

$$\vec{n} \cdot T \vec{\nabla} \mu = -\vec{n} \cdot \vec{v}_0 \quad \text{on } \Gamma_b \quad (11)$$

with $T = \text{diag} \left[\frac{1}{2\alpha_1^2}, \frac{1}{2\alpha_1^2}, \frac{1}{2\alpha_2^2} \right]$. The problem given by (9)-(11), is solved using tetrahedral finite elements (see [4,2]), which leads to a set of 4×4 elemental matrices and 4×1 elemental vectors related to element Ω_e , with $\hat{\psi}_i$ being the form function of the i -th node, $i = 1, 2, 3, 4$, defined in the reference element $\hat{\Omega}_e$ and $|\mathbf{J}|$ the Jacobian of the transformation from Ω_e to $\hat{\Omega}_e$,

$$\begin{aligned} \{\mathbf{A}^e\}_{ij} = & \int_{\hat{\Omega}_e} \left\{ \left(\frac{\partial \hat{\psi}_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{\psi}_i}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \hat{\psi}_i}{\partial \varphi} \frac{\partial \varphi}{\partial x} \right) \left(\frac{\partial \hat{\psi}_j}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{\psi}_j}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \hat{\psi}_j}{\partial \varphi} \frac{\partial \varphi}{\partial x} \right) \right. \\ & + \left(\frac{\partial \hat{\psi}_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{\psi}_i}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \hat{\psi}_i}{\partial \varphi} \frac{\partial \varphi}{\partial y} \right) \left(\frac{\partial \hat{\psi}_j}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{\psi}_j}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \hat{\psi}_j}{\partial \varphi} \frac{\partial \varphi}{\partial y} \right) \\ & \left. + \alpha^2 \left(\frac{\partial \hat{\psi}_i}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial \hat{\psi}_i}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial \hat{\psi}_i}{\partial \varphi} \frac{\partial \varphi}{\partial z} \right) \left(\frac{\partial \hat{\psi}_j}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial \hat{\psi}_j}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial \hat{\psi}_j}{\partial \varphi} \frac{\partial \varphi}{\partial z} \right) \right\} \\ & \cdot |\mathbf{J}| \, d\xi \, d\eta \, d\varphi \end{aligned} \quad (12)$$

$$\begin{aligned} \{\mathbf{b}^e\}_i = & \int_{\hat{\Omega}_e} -2\alpha_1^2 \left\{ u_0 \left(\frac{\partial \hat{\psi}_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{\psi}_i}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \hat{\psi}_i}{\partial \varphi} \frac{\partial \varphi}{\partial x} \right) \right. \\ & + v_0 \left(\frac{\partial \hat{\psi}_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{\psi}_i}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \hat{\psi}_i}{\partial \varphi} \frac{\partial \varphi}{\partial y} \right) \\ & \left. + w_0 \left(\frac{\partial \hat{\psi}_i}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial \hat{\psi}_i}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial \hat{\psi}_i}{\partial \varphi} \frac{\partial \varphi}{\partial z} \right) \right\} \cdot |\mathbf{J}| \, d\xi \, d\eta \, d\varphi \end{aligned} \quad (13)$$

Note that if we set $\varepsilon = \alpha^2$, the elemental matrix may be written as

$$\{\mathbf{A}^e\}_{ij} = \{\mathbf{M}^e\}_{ij} + \varepsilon \{\mathbf{N}^e\}_{ij} \quad (14)$$

The assembling of such matrices yields a symmetric matrix \mathbf{A} given by equation (2). Here we propose to solve the corresponding linear system of equations

(1) by using PCG algorithm. Once we have obtained ϕ , the resulting wind field is computed using equation (8).

3 UPDATING OF THE INCOMPLETE CHOLESKY FACTORIZATION

We will generalize the incomplete factorization proposed by Meurant [6] for the case of matrices $A_\varepsilon = M + \varepsilon D$, with D being diagonal, to matrices $A_\varepsilon = M + \varepsilon N$, with M and N being two $n \times n$ symmetric positive definite matrices that have the same sparsity pattern in this case. We can write A_ε as follows,

$$A_\varepsilon = (m_{ij}) + \varepsilon (n_{ij}) = \begin{pmatrix} m_{11} + \varepsilon n_{11} & (f_{1M} + \varepsilon f_{1N})^T \\ f_{1M} + \varepsilon f_{1N} & M_2 + \varepsilon N_2 \end{pmatrix}$$

where f_{1M}, f_{1N} represent $(n-1) \times 1$ column matrices and M_2, N_2 , $(n-1) \times (n-1)$ matrices.

A factorization of the first row and column of A_ε is carried out,

$$A_\varepsilon = \begin{pmatrix} m_{11} + \varepsilon n_{11} & \mathbf{0} \\ l_{1M} + \varepsilon l_{1N} & \mathbf{I} \end{pmatrix} \begin{pmatrix} (m_{11} + \varepsilon n_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & C_2 \end{pmatrix} \begin{pmatrix} m_{11} + \varepsilon n_{11} & (l_{1M} + \varepsilon l_{1N})^T \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = L_1 Z_1 L_1^T$$

with $l_{1M} = f_{1M}$ and $l_{1N} = f_{1N}$.

Then, identifying term by term, we obtain for matrix C_2 ,

$$C_2 = M_2 + \varepsilon N_2 - \frac{1}{m_{11} + \varepsilon n_{11}} (l_{1M} + \varepsilon l_{1N}) (l_{1M} + \varepsilon l_{1N})^T \quad (15)$$

In order to build the preconditioner, if we only consider the diagonal entries of N as first approximation, then equation (15) is simplified since $l_{1N} = 0$,

$$C_2 = \varepsilon D_2 + M_2 - \frac{1}{m_{11} + \varepsilon n_{11}} l_{1M} l_{1M}^T,$$

An order 0 algorithm is derived from

$$C_2 = \varepsilon D_2 + M_2 - \frac{1}{m_{11}} l_{1M} l_{1M}^T$$

and the entries of C_2 are computed by adding εD_2 to what we would have obtained for the incomplete decomposition of M .

Another approximation consists of considering all the entries in N_2 and neglecting the products εl_{1N} in (15). So, the successive computations of matrices C_i do not involve ε and those matrices may be obtained easily from the M decompositions,

$$C_2 = \varepsilon N_2 + M_2 - \frac{1}{m_{11}} l_{1M} l_{1M}^T$$

and thus, in matrix form,

$$C_2 = \varepsilon N_2 + \begin{pmatrix} m_{22}^{(2)} & f_{2M}^T \\ f_{2M} & M_3 \end{pmatrix} = \begin{pmatrix} m_{22}^{(2)} + \varepsilon n_{22} & (f_{2M} + \varepsilon l_{2N}) \\ f_{2M} + \varepsilon l_{2N} & M_3 + \varepsilon N_3 \end{pmatrix}$$

Only the entries of f_{2M} corresponding to non null entries of M are computed in order to avoid the fill-in, obtaining l_{2M} . So the decomposition of C_2 results in

$$C_2 \approx \begin{pmatrix} m_{22}^{(2)} + \varepsilon n_{22} & \mathbf{0} \\ l_{2M} + \varepsilon l_{2N} & \mathbf{I} \end{pmatrix} \begin{pmatrix} (m_{22}^{(2)} + \varepsilon n_{22})^{-1} & \mathbf{0} \\ \mathbf{0} & C_3 \end{pmatrix} \begin{pmatrix} m_{22}^{(2)} + \varepsilon n_{22} & (l_{2M} + \varepsilon l_{2N})^T \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where, identifying

$$C_3 = M_3 + \varepsilon N_3 - \frac{1}{m_{22}^{(2)} + \varepsilon n_{22}} (l_{2M} + \varepsilon l_{2N}) (l_{2M} + \varepsilon l_{2N})^T$$

Similarly, with the same simplifications, we have,

$$C_3 = M_3 + \varepsilon N_3 - \frac{1}{m_{22}^{(2)}} l_{2M} l_{2M}^T = \begin{pmatrix} m_{33} + \varepsilon n_{33} & (f_{3M} + \varepsilon l_{3N})^T \\ f_{3M} + \varepsilon l_{3N} & M_4 + \varepsilon N_4 \end{pmatrix}$$

which is constructed following the same procedure of C_2 .

In this way, once all the matrices C_i are constructed, the incomplete decomposition of A_ε results in,

$$A_\varepsilon \approx L_1 Z_1 L_1^T = L_1 L_2 Z_2 L_2^T L_1^T = (L_1 L_2 \cdots L_n) Z_n (L_1 L_2 \cdots L_n)^T \quad (16)$$

Z being the diagonal matrix,

$$\begin{pmatrix} (m_{11} + \varepsilon n_{11})^{-1} & & & & & \\ & (m_{22}^{(2)} + \varepsilon n_{22})^{-1} & & & & \\ & & (m_{33}^{(3)} + \varepsilon n_{33})^{-1} & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & (m_{nn}^{(n)} + \varepsilon n_{nn})^{-1} \end{pmatrix}$$

The diagonal entries of the lower triangular matrix $L_1 L_2 \cdots L_n$ are $m_{ii}^{(i)} + \varepsilon n_{ii}$. The respective columns below the diagonal entries are defined by $(n - i) \times 1$ matrices $l_{jM} + \varepsilon l_{jN}$.

4 NUMERICAL EXPERIMENTS

In this section we present the results obtained using PCG with the proposed preconditioners for solving the linear systems of equations arising from the elliptic equation related to a 3-D mass consistent model for wind field adjustment [1,2]. All the experiments were carried out on a XEON Precision 530 with Fortran Double Precision. In the resolution, we always started from the null vector and stopped if $\|r_k\|_2 \leq 10^{-10} \|r_0\|_2$ or if the number of iterations was greater than 10000.

Two cases are presented and two sets of values of parameter α were generated for each of them. The initial population corresponding to the first set with values between 0 and 20, is obtained with the random generator of the GAs package of Levine [10]. It should be as diverse as possible in order to obtain satisfactory results with GAs [11]. For the set with values between 0 and 100, the initial population is generated with a normal distribution. The first case is related to a wind simulation in a region of Gran Canaria Island and we have used an adaptive mesh to produce linear systems of 98999 equations. The second case is also related to a wind simulation in the whole of Gran Canaria Island. We have used another adaptive mesh to produce linear systems of 100643 equations. Both matrices M and N corresponding to the above problems, are SPD.

Figures 1-4 show the behavior of the above mentioned preconditioners. For a wide range of values of α , the timings for reaching convergence are represented in each case. ICHOL_D and ICHOL_N are the ICHOL preconditioners obtained with the two approaches developed in section 3, respectively. These preconditioners are compared with full-ICHOL of matrix A_ε , that is, computing a new

ICHOL decomposition for each ε , and with the use of a single preconditioner, $\text{ICHOL}(A_{\varepsilon_0})$, for all the linear systems.

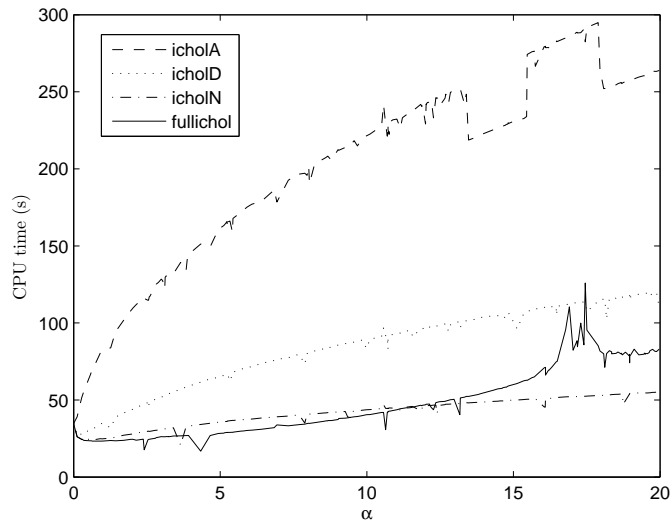


Fig. 1. Windfield.98999: Convergence of PCG with several ICHOL preconditioners for different values of parameter α randomly calculated.

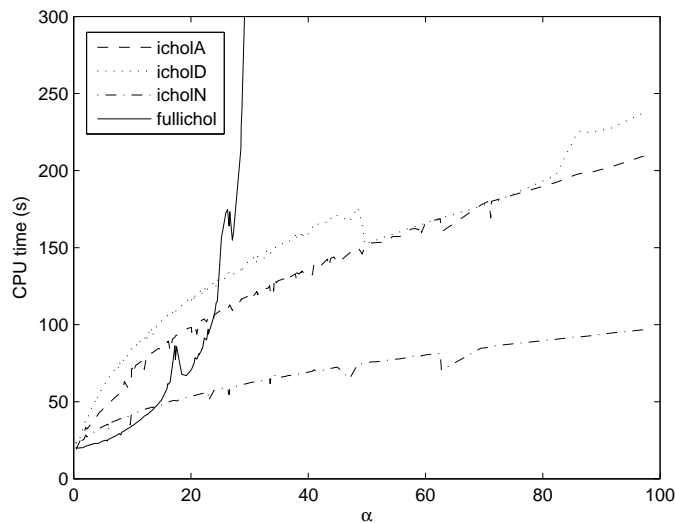


Fig. 2. Windfield.98999: Convergence of PCG with several ICHOL preconditioners for different values of parameter α using normal distribution.

In all the cases, the proposed ICHOL_N preconditioner led to the fastest convergence. For low values of α , the full-ICHOL preconditioner displayed better behavior. However, for higher values of α , it got worse very rapidly and did not even allow convergence to be reached. This result may be explained since we can not always obtain an incomplete factorization ICHOL from the definite positive matrix $(M + \varepsilon N)$ which preserves symmetry and positivity.

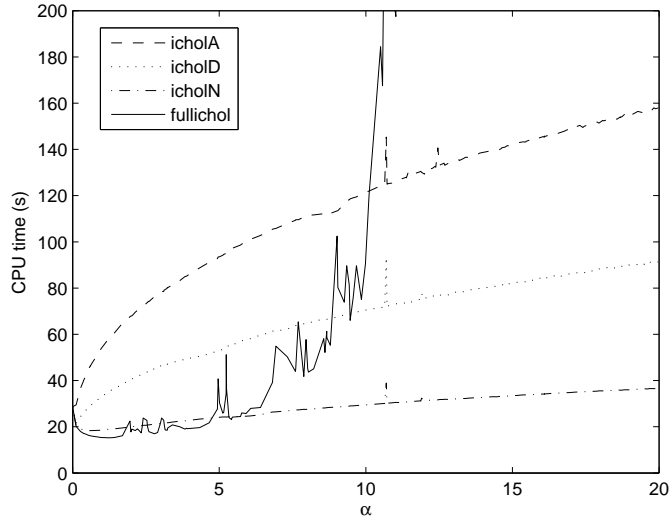


Fig. 3. Windfield.100643: Convergence of PCG with several ICHOL preconditioners for different values of parameter α randomly calculated.

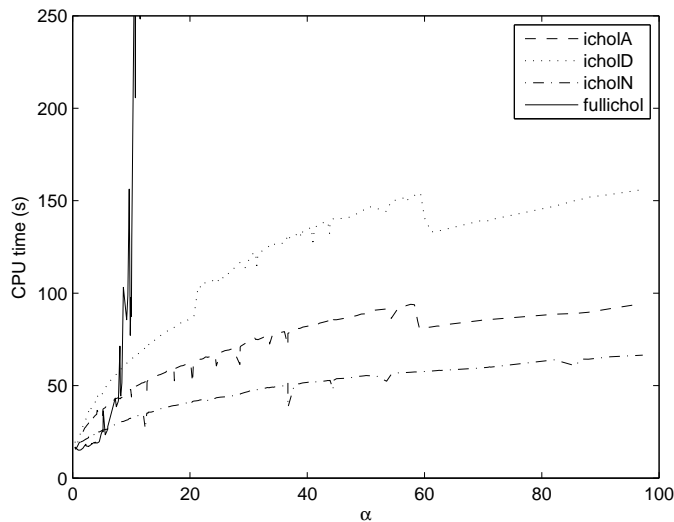


Fig. 4. Windfield.100643: Convergence of PCG with several ICHOL preconditioners for different values of parameter α using normal distribution.

It is known that incomplete factorization may fail when pivots less than or equal to zero, appear [12]; even very small pivots lead to numerical instabilities. In the case of M-matrices or more general H-matrices [13,14] the existence of factorization can be guaranteed but, despite those matrices figuring relatively frequently in various applications, that is not the case that concerns us. Thus, this factorization is not always a suitable preconditioner for Conjugated Gradient method in this wind modeling problems.

However, for the wide range of values of parameter α that are considered

in the two initial populations of the numerical experiments, the proposed preconditioners, ICHOL_D and ICHOL_N , present good results of convergence.

Moreover, an inversion in the behavior of $\text{ICHOL}(A_{\varepsilon_0})$ and ICHOL_D preconditioners can be observed in figures for the two different populations of α used. This inversion can be explained by the fact that the values of the parameter taken to build the preconditioners $\text{ICHOL}(A_{\varepsilon_0})$ are different in both cases.

5 CONCLUSION

We have developed an alternative updating of ICHOL preconditioners in the resolution of linear systems of equations using PCG. The incomplete factorization of matrices of the generalised shifted linear systems arising from numerical simulation of wind fields with mass consistent models may fail as a preconditioner for PCG. However, the proposed ICHOL_N preconditioner leads to better results and becomes a good choice for systems obtained from the application of genetic algorithms.

Further research should be dedicated to updating preconditioners of generalised shifted linear systems with non symmetric matrices.

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