Chapter 1

DUALITY IN COMPLEX STOCHASTIC BOOLEAN SYSTEMS

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Abstract Many different complex systems depend on a large number n of mutually independent random Boolean variables. The most useful representation for these systems –usually called complex stochastic Boolean systems (CSBSs)– is the intrinsic order graph. This is a directed graph on 2^n vertices, corresponding to the 2^n binary n-tuples $(u_1, \ldots, u_n) \in \{0,1\}^n$ of 0s and 1s. In this paper, different duality properties of the intrinsic order graph are rigorously analyzed in detail. The results can be applied to many CSBSs arising from any scientific, technical or social area.

Keywords: Complex stochastic Boolean systems, intrinsic order, intrinsic order graph, complementary n-tuples, duality

1. Introduction

The study of complex systems is at present one of the most relevant research areas in Computer Science and Engineering. In this paper, we focus our attention on the complex stochastic Boolean systems (CSBSs), that is, those complex systems which depend on a certain number n of random Boolean variables. These systems can appear in any knowledge area, since the assumption "random Boolean variables" is satisfied very often in practice.

Using the statistical terminology, a CSBS can be modeled by the n-dimensional Bernoulli distribution. As is well known (see, e.g., [Stuart, et al, 1998]), this distribution consists of n random variables $x_1, \ldots, x_n$, which only take two possible values, 0 or 1, with probabilities

$$\Pr\{x_i = 1\} = p_i, \quad \Pr\{x_i = 0\} = 1 - p_i \quad (1 \leq i \leq n).$$

In the following, we assume that the marginal Bernoulli variables $x_1, \ldots, x_n$ are mutually independent, so that the probability of occurrence of each binary n-tuple, $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, can be computed as the product

$$\Pr\{(u_1, \ldots, u_n)\} = \prod_{i=1}^{n} \Pr\{x_i = u_i\} = \prod_{i=1}^{n} p_i^{u_i} (1 - p_i)^{1-u_i}, \tag{1.1}$$

that is, $\Pr\{(u_1, \ldots, u_n)\}$ is the product of factors p_i if $u_i = 1$, 1-p_i if $u_i = 0$. Throughout this paper, the binary n-tuples $(u_1, \ldots, u_n)$ of 0s and 1s will be also called binary strings or bitstrings, and the parameters $p_1, \ldots, p_n$ of the n-dimensional Bernoulli distribution will be also called basic probabilities.

EXAMPLE 1.1 Let $n = 3$ and $u = (1,0,1) \in \{0,1\}^3$. Let $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.3$. Then, using Eq. (1.1), we have

$$\Pr\{(1,0,1)\} = p_1 (1 - p_2) p_3 = 0.024.$$

One of the most relevant questions in the analysis of CSBSs consists of ordering the binary strings $(u_1, \ldots, u_n)$ according to their occurrence probabilities. Of course, the theoretical and practical interest of this question is obvious. For instance, in [González, 2002; González, et al, 2004] the authors justify the convenience of using binary n-tuples with occurrence probabilities as large as possible, in order to solve, with a low computational cost, some classical problems in Reliability Theory and Risk Analysis.

Of course, computing and ordering all the 2^n binary n-tuple probabilities (in decreasing or increasing order) is only feasible for small values of n. For large values of the number n of basic Boolean variables (the usual situation in practice), we need an alternative strategy. For this purpose, in [González, 2002] we have established a simple, positional criterion that allows one to compare two given binary n-tuple probabilities, $\Pr\{u\}, \Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the n-tuples u, v. We have called it the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it *intrinsically* depends on the positions of the 0s and 1s in the binary strings.

The intrinsic order, denoted by "$\preceq$", is a partial order relation on the set $\{0,1\}^n$ of all binary n-tuples. The usual representation of this kind of binary relations is the Hasse diagram [Stanley, 1997]. In particular, the Hasse diagram of the partially ordered set $(\{0,1\}^n, \preceq)$ is referred to as the *intrinsic order graph* for n variables.

In this context, the main goal of this paper is to state and rigorously prove some properties of the intrinsic order graph. Some of these properties can be found in [González, 2011]. In particular, we focus our attention on several duality properties of this graph. For this purpose, this paper has been organized as follows. In Section 2, we present some previous results on the intrinsic order relation and the intrinsic order graph, enabling non-specialists to follow the paper without difficulty and making the presentation self-contained. Section 3 is devoted to provide different duality properties of the intrinsic order graph. Finally, conclusions are presented in Section 4.

2. The Intrinsic Order Relation and its Graph

The Intrinsic Order Relation

In the context of the CSBSs defined in Section 1, the following simple question arises: Given a certain n-dimensional Bernoulli distribution, how can we order two given binary n-tuples, $u, v \in \{0,1\}^n$, by their occurrence probabilities, without computing them? Of course, the ordering between $\Pr(u)$ and $\Pr(v)$ depends, in general, on the parameters p_i of the Bernoulli distribution, as the following simple example shows.

EXAMPLE 2.1 Let $n = 3,\ u = (0,1,1)$ and $v = (1,0,0)$. Using Eq. (1.1) for $p_1 = 0.1,\ p_2 = 0.2,\ p_3 = 0.3$, we have:

$$\Pr\{(0,1,1)\} = 0.054 < \Pr\{(1,0,0)\} = 0.056,$$

while for $p_1 = 0.2,\ p_2 = 0.3,\ p_3 = 0.4$, we have:

$$\Pr\{(0,1,1)\} = 0.096 > \Pr\{(1,0,0)\} = 0.084.$$

However, for some pairs of binary strings, the ordering between their occurrence probabilities is independent of the basic probabilities p_i, and it only depends on the relative positions of their 0s and 1s. More precisely, the following theorem [González, 2002; González, 2003] provides us with an intrinsic order criterion –denoted from now on by the acronym IOC– to compare the occurrence probabilities of two given n-tuples of 0s 1s without computing them.

THEOREM 2.2 *Let $n \geq 1$. Let $x_1, \ldots, x_n$ be n mutually independent Bernoulli variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy*

$$0 < p_1 \leq p_2 \leq \ldots \leq p_n \leq \frac{1}{2}. \tag{2.1}$$

Then the probability of the n-tuple $v = (v_1, \ldots, v_n) \in \{0,1\}^n$ is intrinsically less than or equal to the probability of the n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ (that is, for all set $\{p_i\}_{i=1}^n$satisfying (2.1)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

REMARK 2.3 In the following, we assume that the parameters p_i always satisfy condition (2.1). Note that this hypothesis is not restrictive for practical applications because, if for some $i : p_i > 0.5$, then we only need to consider the variable $\overline{x_i} = 1 - x_i$, instead of x_i. Next, we order the n Bernoulli variables by increasing order of their probabilities.

REMARK 2.4 The $\binom{0}{1}$ column preceding to each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

REMARK 2.5 The term *corresponding,* used in Theorem 2.2, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding to each other. In other words: For each $\binom{1}{0}$ column in matrix M_v^u, the number of preceding $\binom{0}{1}$ columns must be strictly greater than the number of preceding $\binom{1}{0}$ columns.

REMARK 2.6 IOC can be equivalently reformulated in the following way, involving only the 1-bits of u and v (with no need to use their 0-bits). Matrix M_v^u satisfies IOC if and only if either u has no 1-bits (i.e., u is the zero n-tuple) or for each 1-bit in u there exists (at least) one corresponding 1-bit in v placed at the same or at a previous position. In other words, either u has no 1-bits or for each 1-bit in u, say $u_i = 1$, the number of 1-bits in $(v_1, \ldots, v_i)$ must be greater than or equal to the number of 1-bits in $(u_1, \ldots, u_i)$.

The matrix condition IOC, stated by Theorem 2.2 or by Remark 2.6, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it only depends on the relative positions of the

0s and 1s in the binary n-tuples u, v. Theorem 2.2 naturally leads to the following partial order relation on the set $\{0,1\}^n$ [González, 2003]. The so-called intrinsic order will be denoted by "$\preceq$", and we shall write $u \succeq v$ ($u \preceq v$) to indicate that u is intrinsically greater (less) than or equal to v. The partially ordered set (from now on, poset, for short) $(\{0,1\}^n, \preceq)$ on n Boolean variables, will be denoted by I_n.

DEFINITION 2.7 *For all $u, v \in \{0,1\}^n$*

$$\begin{aligned} v \preceq u \ \text{ iff } \ & \Pr\{v\} \leq \Pr\{u\} \ \text{ for all set } \ \{p_i\}_{i=1}^n \ \text{ s.t. (2.1)} \\ \text{ iff } \ & M_v^u \text{ satisfies IOC.} \end{aligned}$$

EXAMPLE 2.8 Neither $(0,1,1) \preceq (1,0,0)$, nor $(1,0,0) \preceq (0,1,1)$ because the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

do not satisfy IOC (Remark 2.5). Therefore $(0,1,1)$ and $(1,0,0)$ are incomparable by intrinsic order, i.e., the ordering between $\Pr\{(0,1,1)\}$ and $\Pr\{(1,0,0)\}$ depends on the basic probabilities p_i, as Example 2.1 has shown.

EXAMPLE 2.9 $(1,1,0,1,0,0) \preceq (0,0,1,1,0,1)$ because matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

satisfies IOC (Remark 2.4). Thus, for all $\{p_i\}_{i=1}^6$ s.t. (2.1)

$$\Pr\{(1,1,0,1,0,0)\} \leq \Pr\{(0,0,1,1,0,1)\}.$$

EXAMPLE 2.10 For all $n \geq 1$, the binary n-tuples

$$\left(0, \overset{n}{\ldots}, 0\right) \equiv 0 \quad \text{and} \quad \left(1, \overset{n}{\ldots}, 1\right) \equiv 2^n - 1$$

are the maximum and minimum elements, respectively, in the poset I_n. Indeed, both matrices

$$\begin{pmatrix} 0 & \ldots & 0 \\ u_1 & \ldots & u_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1 & \ldots & u_n \\ 1 & \ldots & 1 \end{pmatrix}$$

satisfy IOC, since they have no $\binom{1}{0}$ columns!
Thus, for all $u \in \{0,1\}^n$ and for all $\{p_i\}_{i=1}^n$ s.t. (2.1)

$$\Pr\left\{\left(1, \overset{n}{\ldots}, 1\right)\right\} \leq \Pr\{(u_1, \ldots, u_n)\} \leq \Pr\left\{\left(0, \overset{n}{\ldots}, 0\right)\right\}.$$

Many different properties of the intrinsic order can be immediately derived from its simple matrix description IOC [González, 2002; González, 2003; González, 2007]. For instance, denoting by $w_H(u)$ the Hamming weight –or weight, simply– of u (i.e., the number of 1-bits in u), by $u_{(10}$ the decimal representation of u, and by $\leq_{lex}$ the usual lexicographic (truth-table) order on $\{0,1\}^n$, i.e.,

$$w_H(u) := \sum_{i=1}^{n} u_i, \quad u_{(10} := \sum_{i=1}^{n} 2^{n-i} u_i, \quad u \leq_{lex} v \text{ iff } u_{(10} \leq v_{(10}$$

then we have the following two necessary (but not sufficient) conditions for intrinsic order (see [González, 2003] for the proof).

Corollary 2.11 *For all $n \geq 1$ and for all $u, v \in \{0,1\}^n$*

$$u \succeq v \Rightarrow w_H(u) \leq w_H(v),$$
$$u \succeq v \Rightarrow u_{(10} \leq v_{(10}.$$

A Hasse Diagram: The Intrinsic Order Graph

Now, the graphical representation of the poset $I_n = (\{0,1\}^n, \preceq)$ is presented. The usual representation of a poset is its Hasse diagram (see [Stanley, 1997] for more details about these diagrams). Specifically, for our poset I_n, its Hasse diagram is a directed graph (digraph, for short) whose vertices are the 2^n binary n-tuples of 0s and 1s, and whose edges go upward from v to u whenever u covers v, denoted by $u \vartriangleright v$. This means that u is intrinsically greater than v with no other elements between them, i.e.,

$$u \vartriangleright v \quad \Leftrightarrow \quad u \succ v \text{ and } \nexists\ w \in \{0,1\}^n \quad \text{s.t.} \quad u \succ w \succ v.$$

A simple matrix characterization of the covering relation for the intrinsic order is given in the next theorem; see [González, 2006] for the proof.

Theorem 2.12 (Covering relation in I_n) *Let $n \geq 1$ and let $u, v \in \{0,1\}^n$. Then $u \vartriangleright v$ if and only if the only columns of matrix M_v^u different from $\binom{0}{0}$ and $\binom{1}{1}$ are either its last column $\binom{0}{1}$ or just two columns, namely one $\binom{1}{0}$ column immediately preceded by one $\binom{0}{1}$ column, i.e., either*

$$M_v^u = \begin{pmatrix} u_1 & \ldots & u_{n-1} & 0 \\ u_1 & \ldots & u_{n-1} & 1 \end{pmatrix} \tag{2.2}$$

or there exists i $(2 \leq i \leq n)$ s.t.

$$M_v^u = \begin{pmatrix} u_1 & \ldots & u_{i-2} & 0 & 1 & u_{i+1} & \ldots & u_n \\ u_1 & \ldots & u_{i-2} & 1 & 0 & u_{i+1} & \ldots & u_n \end{pmatrix}. \tag{2.3}$$

EXAMPLE 2.13 For $n = 4$, we have

$$6 \rhd 7 \text{ since } M_7^6 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \text{ has the pattern (2.2),}$$

$$10 \rhd 12 \text{ since } M_{12}^{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ has the pattern (2.3).}$$

The Hasse diagram of the poset I_n will be also called the *intrinsic order graph* for n variables, denoted as well by I_n.

For small values of n, the intrinsic order graph I_n can be directly constructed by using either Theorem 2.2 (matrix description of the intrinsic order) or Theorem 2.12 (matrix description of the covering relation for the intrinsic order). For instance, for $n = 1$: $I_1 = (\{0, 1\}, \preceq)$, and its Hasse diagram is shown in Figure 1.1.

$$\begin{array}{c} 0 \\ | \\ 1 \end{array}$$

Figure 1.1. The intrinsic order graph for $n = 1$.

Indeed I_1 contains a downward edge from 0 to 1 because (see Theorem 2.2) $0 \succ 1$, since matrix $\binom{0}{1}$ has no $\binom{1}{0}$ columns! Alternatively, using Theorem 2.12, we have that $0 \rhd 1$, since matrix $\binom{0}{1}$ has the pattern (2.2)! Moreover, this is in accordance with the obvious fact that

$$\Pr\{0\} = 1 - p_1 \geq p_1 = \Pr\{1\}, \text{ since } \ p_1 \leq 1/2 \text{ due to Eq. (2.1)!}$$

However, for large values of n, a more efficient method is needed. For this purpose, in [González, 2006] the following algorithm for iteratively building up I_n (for all $n \geq 2$) from I_1 (depicted in Figure 1.1), has been developed.

THEOREM 2.14 (BUILDING UP I_n FROM I_1) *Let $n \geq 2$. The graph of the poset $I_n = \{0, \ldots, 2^n - 1\}$ (on 2^n nodes) can be drawn simply by adding to the graph of the poset $I_{n-1} = \left\{0, \ldots, 2^{n-1} - 1\right\}$ (on 2^{n-1} nodes) its isomorphic copy $2^{n-1} + I_{n-1} = \left\{2^{n-1}, \ldots, 2^n - 1\right\}$ (on 2^{n-1} nodes). This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of I_n. Finally, the edges connecting one vertex u of I_{n-1} with the other vertex v of $2^{n-1} + I_{n-1}$ are given by the set of 2^{n-2} vertex pairs*

$$\left\{(u, v) \equiv \left(u_{(10}, 2^{n-2} + u_{(10}\right) \;\middle|\; 2^{n-2} \leq u_{(10} \leq 2^{n-1} - 1\right\}.$$

Figure 1.2 illustrates the above iterative process for the first few values of n, denoting all the binary n-tuples by their decimal equivalents. Basically, we first add to I_{n-1} its isomorphic copy $2^{n-1} + I_{n-1}$. This addition must be performed by placing the powers of two, 2^{n-2} and 2^{n-1}, at consecutive levels in the intrinsic order graph. The reason is simply that

$$2^{n-2} \triangleright 2^{n-1} \quad \text{since matrix } M_{2^{n-1}}^{2^{n-2}} \quad \text{has the pattern (2.3).}$$

Then, we connect one-to-one the nodes of "the second half of the first half" to the nodes of "the first half of the second half": A nice fractal property of I_n!

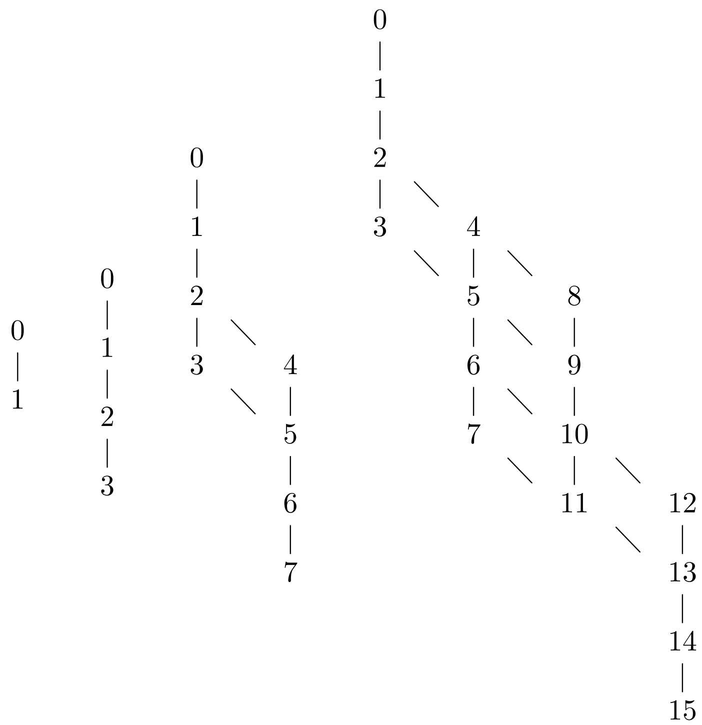

Figure 1.2. The intrinsic order graphs for $n = 1, 2, 3, 4$.

Each pair (u, v) of vertices connected in I_n either by one edge or by a longer path, descending from u to v, means that u is intrinsically greater than v, i.e., $u \succ v$. On the contrary, each pair (u, v) of non-connected vertices in I_n either by one edge or by a longer descending path, means that u and v are incomparable by intrinsic order, i.e., $u \not\succ v$ and $v \not\succ u$.

The edgeless graph for a given graph is obtained by removing all its edges, keeping its nodes at the same positions. In Figures 1.3 & 1.4, the edgeless intrinsic order graphs of I_5 & I_6, respectively, are depicted.

0							
1							
2							
3	4						
	5	8					
	6	9		16			
	7	10		17			
		11	12	18			
			13	19	20		
			14		21	24	
			15		22	25	
					23	26	
						27	28
							29
							30
							31

Figure 1.3. The edgeless intrinsic order graph for $n = 5$.

0															
1															
2															
3	4														
	5	8													
	6	9		16											
	7	10		17				32							
		11	12	18				33							
			13	19	20			34							
			14		21	24		35	36						
			15		22	25			37	40					
					23	26			38	41		48			
						27	28		39	42		49			
							29			43	44	50			
							30				45	51	52		
							31				46		53	56	
											47		54	57	
													55	58	
														59	60
															61
															62
															63

Figure 1.4. The edgeless intrinsic order graph for $n = 6$.

For further theoretical properties and practical applications of the intrinsic order and the intrinsic order graph, we refer the reader to e.g., [González, 2002; González, 2006; González, 2007; González, 2010; González, 2011; González, et al, 2004].

3. Duality Properties in the Intrinsic Order Graph

First, we need to set the following nomenclature and notation.

DEFINITION 3.1 *The complementary n-tuple u^c of a given binary n-tuple $u = (u_1, \dots, u_n) \in \{0,1\}^n$ is obtained by changing its 0s by 1s and its 1s by 0s*

$$u^c = (u_1, \dots, u_n)^c = (1 - u_1, \dots, 1 - u_n) .$$

The complementary set S^c of a given subset $S \subseteq \{0,1\}^n$ of binary n-tuples is the set of the complementary n-tuples of all the n-tuples of S

$$S^c = \{u^c \mid u \in S\} .$$

REMARK 3.2 Note that for all $u = (u_1, \dots, u_n) \in \{0,1\}^n$ and for all $S, T \subseteq \{0,1\}^n$, we obviously have

$$\text{(i)}\ \ (u^c)^c = u, \ \ \text{(ii)}\ \ (S^c)^c = S, \ \ \text{(iii)}\ \ u \in S \Leftrightarrow u^c \in S^c,$$

$$\text{(iv)}\ \ (S \cup T)^c = S^c \cup T^c, \ \ \text{(v)}\ \ w_H(u) + w_H(u^c) = n.$$

The following proposition states a duality property of the intrinsic order, that explains the symmetric structure of the intrinsic order graph.

PROPOSITION 3.3 *Let $n \geq 1$ and $u, v \in \{0,1\}^n$. Then*

$$u \rhd v \ \Leftrightarrow \ v^c \rhd u^c,$$
$$u \succeq v \ \Leftrightarrow \ v^c \succeq u^c.$$

Proof. Clearly, the $\binom{0}{0}$, $\binom{1}{1}$, $\binom{0}{1}$ and $\binom{1}{0}$ columns in matrix M_v^u, respectively become $\binom{1}{1}$, $\binom{0}{0}$, $\binom{0}{1}$ and $\binom{1}{0}$ columns in matrix $M_{u^c}^{v^c}$.
Hence, on one hand, using Theorem 2.12, we have that $u \rhd v$ iff matrix M_v^u has either the pattern (2.2) or the pattern (2.3) iff matrix $M_{u^c}^{v^c}$ has either the pattern (2.2) or the pattern (2.3), respectively, iff $v^c \rhd u^c$.
On the other hand, using Theorem 2.2, we have that $u \succeq v$ iff matrix M_v^u satisfies IOC iff matrix $M_{u^c}^{v^c}$ satisfies IOC iff $v^c \succeq u^c$. □

Next corollary provides us with two easy criteria for rapidly identifying pairs of complementary binary strings in the intrinsic order graph.

Corollary 3.4 *Let $n \geq 1$ and $u, v \in \{0,1\}^n$. Then u and v are complementary n-tuples if and only if their decimal equivalents sum up to $2^n - 1$ if and only if they are placed at symmetric positions (with respect to the central point) in the (edgeless) graph I_n.*

Proof. Using Definition 3.1, we have that u and v are complementary n-tuples if and only if

$$u + v = \left(1, \overset{n}{\ldots}, 1\right) \equiv 2^n - 1.$$

Using Proposition 3.3, we have that u and v are complementary n-tuples if and only if they are placed at symmetric positions (with respect to the central point) in the (edgeless) graph I_n. □

Hence, the simplest way to verify that two binary n-tuples are complementary, when we use their decimal representations, is to check that they sum up to $2^n - 1$.

Example 3.5 The binary 5-tuples $6 \equiv (0,0,1,1,0)$ & $25 \equiv (1,1,0,0,1)$ are complementary, since $6 + 25 = 31 = 2^5 - 1$. Alternatively, we can see that 6 and 25 are placed at symmetric positions (with respect to the central point) in the edgeless graph I_5, depicted in Figure 1.3.

Example 3.6 The complementary 6-tuple of the binary 6-tuple $50 \equiv (1,1,0,0,1,0)$ is $13 \equiv (0,0,1,1,0,1)$, since $\left(2^6 - 1\right) - 50 = 63 - 50 = 13$. Alternatively, we can see that 13 is the symmetric node (with respect to the central point) of 50 in the edgeless graph I_6, depicted in Figure 1.4.

Many different consequences can be derived from Proposition 3.3. Some of them are presented in the following corollaries. Before each of them we give some definitions required to understand the statements of the corollaries.

Definition 3.7 *For every binary n-tuple $u \in \{0,1\}^n$, the set C^u (the set C_u, respectively) is the set of all binary n-tuples v whose occurrence probabilities $\Pr\{v\}$ are always less (greater, respectively) than or equal to $\Pr\{u\}$, i.e., according to Definition 2.7, those n-tuples v intrinsically less (greater, respectively) than or equal to u, i.e.,*

$$C^u = \{v \in \{0,1\}^n \mid u \succeq v\},$$

$$C_u = \{v \in \{0,1\}^n \mid v \succeq u\}.$$

DEFINITION 3.8 *For every binary n-tuple $u \in \{0,1\}^n$, $Inc\,(u)$ is the set of all binary n-tuples v intrinsically incomparable with u, i.e.,*

$$Inc\,(u) = \{v \in \{0,1\}^n \mid u \not\succeq v, \;\; u \not\preceq v\} = \{0,1\}^n - (C^u \cup C_u)\,.$$

COROLLARY 3.9 *For all $n \geq 1$ and for all $u \in \{0,1\}^n$, we have*

(i) $(C^u)^c = C_{u^c}$, (ii) $(C_u)^c = C^{u^c}$, (iii) $(Inc\,(u))^c = Inc\,(u^c)$.

Proof. To prove (i) it suffices to use Remark 3.2, Proposition 3.3 and Definition 3.7. Indeed

$$v \in (C^u)^c \Leftrightarrow v^c \in C^u \Leftrightarrow u \succeq \; v^c \Leftrightarrow v \succeq \; u^c \Leftrightarrow v \in C_{u^c}.$$

Clearly (ii) is equivalent to (i); see Remark 3.2. Finally, to prove (iii), we use (i), (ii), Remark 3.2 and Definition 3.8

$$\begin{aligned} v \in (Inc\,(u))^c &\Leftrightarrow v^c \in Inc\,(u) \Leftrightarrow v^c \notin (C^u \cup C_u) \\ &\Leftrightarrow v \notin (C^u \cup C_u)^c \Leftrightarrow v \notin C_{u^c} \cup C^{u^c} \Leftrightarrow v \in Inc\,(u^c)\,, \end{aligned}$$

as was to be shown. □

The following definition (see [Stanley, 1997]) deals with the general theory of posets.

DEFINITION 3.10 *Let $(P, \leq)$ be a poset and $u \in P$. Then*
(i) The lower shadow of u is the set

$$\Delta\,(u) = \{v \in P \mid v \text{ is covered by } u\} = \{v \in P \mid u \rhd v\}\,.$$

(ii) The upper shadow of u is the set

$$\nabla\,(u) = \{v \in P \mid v \text{ covers } u\} = \{v \in P \mid v \rhd u\}\,.$$

Particularly, for our poset $P = I_n$, on one hand, regarding the lower shadow of $u \in \{0,1\}^n$, using Theorem 2.12, we have

$$\begin{aligned} \Delta\,(u) &= \{v \in \{0,1\}^n \mid u \rhd v\} \\ &= \{v \in \{0,1\}^n \mid M_v^u \text{ has either the pattern (2.2) or (2.3)}\}\,. \end{aligned}$$

and, on the other hand, regarding the upper shadow of $u \in \{0,1\}^n$, using again Theorem 2.12, we have

$$\begin{aligned} \nabla\,(u) &= \{v \in \{0,1\}^n \mid v \rhd u\} \\ &= \{v \in \{0,1\}^n \mid M_u^v \text{ has either the pattern (2.2) or (2.3)}\}\,. \end{aligned}$$

COROLLARY 3.11 *For all $n \geq 1$ and for all $u \in \{0,1\}^n$, we have*

$$\text{(i) } \Delta(u) = \nabla^c(u^c), \qquad \text{(ii) } \nabla(u) = \Delta^c(u^c).$$

Proof. To prove (i), we use Remark 3.2, Proposition 3.3 and Definition 3.10. Indeed,

$$v \in \Delta(u) \Leftrightarrow u \rhd v \Leftrightarrow v^c \rhd u^c \Leftrightarrow v^c \in \nabla(u^c) \Leftrightarrow v \in \nabla^c(u^c).$$

Clearly (ii) is equivalent to (i); see Remark 3.2. □

The following definition (see [Diestel, 2005]) deals with the general theory of graphs.

DEFINITION 3.12 *The neighbors of a given vertex u in a graph, are all those nodes adjacent to u (i.e., connected by one edge to u). The degree of a given vertex u –denoted by $\delta(u)$– in a graph is the number of neighbors of u.*

In particular, for (the cover graph of) a Hasse diagram, the neighbors of vertex u either cover u or are covered by u. In other words, the set $N(u)$ of neighbors of a vertex u is the union of its lower and upper shadows, i.e.,

$$N(u) = \Delta(u) \cup \nabla(u), \quad \delta(u) = |N(u)| = |\Delta(u)| + |\nabla(u)|. \qquad (3.1)$$

COROLLARY 3.13 *For all $n \geq 1$ and for all $u \in \{0,1\}^n$, the sets of neighbors of u and u^c are complementary. In particular, any two complementary n-tuples u and u^c have the same degree.*

Proof. Using Remark 3.2, Corollary 3.11, Definition 3.12 and Eq. (3.1), we have

$$\begin{aligned} N^c(u) &= [\Delta(u) \cup \nabla(u)]^c = \Delta^c(u) \cup \nabla^c(u) \\ &= \nabla(u^c) \cup \Delta(u^c) = N(u^c) \end{aligned}$$

and consequently

$$\delta(u) = |N(u)| = |N^c(u)| = |N(u^c)| = \delta(u^c),$$

as was to be shown. □

4. Conclusions

The analysis of CSBSs can be performed by using the intrinsic ordering between binary n-tuples of 0s and 1s. The duality property of the intrinsic order relation for complementary n-tuples (obtained by changing 0s into 1s and 1s into 0s), implies many different properties of CSBSs. Some of these properties has been rigorously proved and illustrated by the intrinsic order graph.

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