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Complex Stochastic Boolean Systems: New Properties of the Intrinsic Order Graph

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***Abstract*—A complex stochastic Boolean system (CSBS) is a system depending on an arbitrary number n of stochastic Boolean variables. The analysis of CSBSs is mainly based on the intrinsic order: a partial order relation defined on the set $\{0,1\}^n$ of binary n-tuples. The usual graphical representation for a CSBS is the intrinsic order graph: the Hasse diagram of the intrinsic order. In this paper, some new properties of the intrinsic order graph are studied. Particularly, the set and the number of its edges, the degree and neighbors of each vertex, as well as typical properties, such as the symmetry and fractal structure of this graph, are analyzed.**

***Index Terms*—complex stochastic Boolean system, Hasse diagram, intrinsic order, intrinsic order graph, poset.**

I. Introduction

In many different scientific, technical or social areas, one can find phenomena depending on an arbitrarily large number n of random Boolean variables. In other words, the n basic variables of the system are assumed to be stochastic and they only take two possible values: either 0 or 1. We call such a system: a *complex stochastic Boolean system* (CSBS). Each one of the 2^n possible elementary states associated to a CSBS is given by a binary n-tuple $u=(u_1,\ldots,u_n)\in\{0,1\}^n$ of 0s and 1s, and it has its own occurrence probability $\Pr\{(u_1,\ldots,u_n)\}$.

Using the statistical terminology, a CSBS on n variables $x_1,\ldots,x_n$ can be modeled by the n-dimensional Bernoulli distribution with parameters $p_1,\ldots,p_n$ defined by

$$\Pr\{x_i=1\}=p_i,\ \Pr\{x_i=0\}=1-p_i,$$

Throughout this paper we assume that the n Bernoulli variables x_i are mutually statistically independent, so that the occurrence probability of a given binary string of length n, $u=(u_1,\ldots,u_n)\in\{0,1\}^n$, can be easily computed as

$$\Pr\{u\}=\prod_{i=1}^{n}p_i^{u_i}\left(1-p_i\right)^{1-u_i}, \tag{1}$$

that is, $\Pr\{u\}$ is the product of factors p_i if $u_i=1$, $1-p_i$ if $u_i=0$.

Example 1.1: Let $n=4$ and $u=(0,1,0,1)\in\{0,1\}^4$. Let $p_1=0.1$, $p_2=0.2$, $p_3=0.3$, $p_4=0.4$. Then using (1), we have

$$\Pr\{(0,1,0,1)\}=(1-p_1)\,p_2\,(1-p_3)\,p_4=0.0504.$$

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The behavior of a CSBS is determined by the ordering between the current values of the 2^n associated binary n-tuple probabilities $\Pr\{u\}$. Computing all these 2^n probabilities –by using (1)– and ordering them in decreasing or increasing order of their values is only possible in practice for small values of the number n of basic variables. However, for large values of n, to overcome the exponential nature of this problem, we need alternative procedures for comparing the binary string probabilities. For this purpose, in [2] we have defined a partial order relation on the set $\{0,1\}^n$ of all the 2^n binary n-tuples, the so-called *intrinsic order* between binary n-tuples.

The intrinsic order relation is characterized by a simple positional criterion, the so-called *intrinsic order criterion* (IOC). IOC enables one to compare (to order) two given binary n-tuple probabilities $\Pr\{u\},\Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the binary n-tuples u,v.

The most useful graphical representation of a CSBS is the intrinsic order graph. This is a symmetric, self-dual diagram on 2^n nodes (denoted by I_n) that displays all the binary n-tuples from top to bottom in decreasing order of their occurrence probabilities. Formally, I_n is the Hasse diagram of the intrinsic partial order relation on $\{0,1\}^n$.

In this context, the main goal of this paper is to present some new properties of the intrinsic order graph. In particular, we give the set and the number of edges of I_n, the set and the number of elements which are neighbors (adjacent) in the graph to a fixed binary n-tuple $u\in\{0,1\}^n$, and analyze the properties of symmetry and fractal character of I_n.

For this purpose, this paper has been organized as follows. In Section II, we present some preliminaries about the intrinsic order and the intrinsic order graph, to make this paper self-contained. Section III is devoted to present the new properties of the intrinsic order graph. Finally, in Section IV, we present our conclusions.

II. Background in Intrinsic Order

Throughout this paper, we indistinctly denote the n-tuple $u\in\{0,1\}^n$ by its binary representation $(u_1,\ldots,u_n)$ or by its decimal representation, and we use the symbol “ $\equiv$” to indicate the conversion between these two numbering systems. The decimal numbering and the Hamming weight (i.e., the number of 1-bits) of u will be respectively denoted by

$$u\equiv u_{(10}=\sum_{i=1}^{n}2^{n-i}u_i,\quad w_H(u)=\sum_{i=1}^{n}u_i.$$

Given two binary n-tuples $u,v\in\{0,1\}^n$, the ordering between their occurrence probabilities $\Pr(u)$, $\Pr(v)$ obviously depends on the Bernoulli parameters p_i, as the following simple example shows.

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Example 2.1: Let $n = 3$, $u = (0,1,1)$ and $v = (1,0,0)$. For $p_1 = 0.1,\ p_2 = 0.2,\ p_3 = 0.3$, using (1), we have:

$$\Pr\{(0,1,1)\} = 0.054 < \Pr\{(1,0,0)\} = 0.056,$$

for $p_1 = 0.2,\ p_2 = 0.3,\ p_3 = 0.4$, using (1), we have:

$$\Pr\{(0,1,1)\} = 0.096 > \Pr\{(1,0,0)\} = 0.084.$$

However, as mentioned in Section I, in [2] we have established an intrinsic, positional criterion to compare the occurrence probabilities of two given binary n-tuples without computing them. This criterion is presented in detail in Section II-*A*, while its graphical representation is shown in Section II-*B*.

A. The Intrinsic Order Criterion

Theorem 2.1 (The intrinsic order theorem): Let $n \geq 1$. Let $x_1, \ldots, x_n$ be n mutually independent Bernoulli variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy

$$0 < p_1 \leq p_2 \leq \cdots \leq p_n \leq 0.5. \qquad (2)$$

Then the occurrence probability of the binary n-tuple v, i.e., $v = (v_1, \ldots, v_n) \in \{0,1\}^n$, is *intrinsically* less than or equal to the occurrence probability of the binary n-tuple u, i.e., $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (2)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 2.1: In the following, we assume that the parameters p_i always satisfy condition (2). Fortunately, this hypothesis is not restrictive for practical applications.

Remark 2.2: The $\binom{0}{1}$ column preceding each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

Remark 2.3: The term *corresponding*, used in Theorem 2.1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding each other. In other words, for each $\binom{1}{0}$ column in matrix M_v^u the number of preceding $\binom{0}{1}$ columns must be strictly greater than the number of preceding $\binom{1}{0}$ columns.

Claim 2.1: IOC can be equivalently reformulated in the following way, involving only the 1-bits of u and v (with no need to use their 0-bits). Matrix M_v^u satisfies IOC if and only if either u has no 1-bits (i.e., u is the zero n-tuple) or for each 1-bit in u there exists (at least) one corresponding 1-bit in v placed at the same or at a previous position. In other words, either u has no 1-bits or for each 1-bit in u, say $u_i = 1$, the number of 1-bits in $(v_1, \ldots, v_i)$ must be greater than or equal to the number of 1-bits in $(u_1, \ldots, u_i)$.

The matrix condition IOC, stated by Theorem 2.1 or by Claim 2.1, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it only depends on the relative positions of the 0s and 1s in the binary strings u and v. Theorem 2.1 naturally leads to the following partial order relation on the set $\{0,1\}^n$ [2], [3]. The so-called intrinsic order will be denoted by "$\preceq$", and when $v \preceq u$ we say that v *is intrinsically less than or equal to* u (or u *is intrinsically greater than or equal to* v).

Definition 2.1: For all $u, v \in \{0,1\}^n$

$$v \preceq u \ \text{ iff } \ \Pr\{v\} \leq \Pr\{u\} \ \text{ for all set } \{p_i\}_{i=1}^n \text{ s.t. (2)}$$

$$\text{iff matrix } M_v^u \text{ satisfies IOC.}$$

In the following, the partially ordered set (poset, for short) for n variables $(\{0,1\}^n, \preceq)$ will be denoted by I_n; see [10] for more details about posets.

Example 2.2: For $n = 3$:

$$3 \equiv (0,1,1) \npreceq (1,0,0) \equiv 4 \ \& \ (1,0,0) \npreceq (0,1,1) \ \text{ since}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

do not satisfy IOC (Remark 2.3). Therefore, $(0,1,1)$ and $(1,0,0)$ are incomparable by intrinsic order, i.e., the ordering between $\Pr\{(0,1,1)\}$ and $\Pr\{(1,0,0)\}$ depends on the basic probabilities p_i, as Example 2.1 has shown.

Example 2.3: For $n = 4$:

$$12 \equiv (1,1,0,0) \preceq (0,0,1,1) \equiv 3 \ \text{ since}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

satisfies IOC (Remark 2.2). For all $0 < p_1 \leq \cdots \leq p_4 \leq \frac{1}{2}$

$$\Pr\{(1,1,0,0)\} \leq \Pr\{(0,0,1,1)\}.$$

B. The Intrinsic Order Graph

In this subsection, the graphical representation of the poset $I_n = (\{0,1\}^n, \preceq)$ is presented. The usual representation of a poset is its Hasse diagram (see [10] for more details about these diagrams). Specifically, for our poset I_n, its Hasse diagram is a directed graph (digraph, for short) whose vertices are the 2^n binary n-tuples of 0s and 1s, and whose edges go upward from v to u whenever u covers v, denoted by $u \triangleright v$. This means that u is intrinsically greater than v with no other elements between them, i.e.,

$$u \triangleright v \quad \Leftrightarrow \quad u \succ v \text{ and } \nexists \ w \in \{0,1\}^n \quad \text{s.t.} \quad u \succ w \succ v.$$

A simple matrix characterization of the covering relation for the intrinsic order is given in the next theorem; see [4] for the proof.

Theorem 2.2 (Covering relation in I_n): Let $n \geq 1$ and $u, v \in \{0,1\}^n$. Then $u \triangleright v$ if and only if the only columns of matrix M_v^u different from $\binom{0}{0}$ and $\binom{1}{1}$ are either its last column $\binom{0}{1}$ or just two columns, namely one $\binom{1}{0}$ column immediately preceded by one $\binom{0}{1}$ column, i.e., either

$$M_v^u = \begin{pmatrix} u_1 & \ldots & u_{n-1} & 0 \\ u_1 & \ldots & u_{n-1} & 1 \end{pmatrix} \ \text{or} \qquad (3)$$

$$M_v^u = \begin{pmatrix} u_1 & \ldots & u_{i-2} & 0 & 1 & u_{i+1} & \ldots & u_n \\ u_1 & \ldots & u_{i-2} & 1 & 0 & u_{i+1} & \ldots & u_n \end{pmatrix}. \qquad (4)$$

$$(2 \leq i \leq n)$$

Example 2.4: For $n = 4$, we have

$$6 \triangleright 7 \ \text{ since } \ M_7^6 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \ \text{ has the pattern (3),}$$

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$10 \rhd 12$ since $M_{12}^{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ has the pattern (4).

The Hasse diagram of the poset I_n will be also called the *intrinsic order graph* for n variables, denoted as well by I_n.

For small values of n, the intrinsic order graph I_n can be directly constructed by using either Theorem 2.1 or Theorem 2.2. For instance, for $n = 1$: $I_1 = (\{0, 1\}, \preceq)$, and its Hasse diagram is shown in Fig. 1.

Fig. 1. The intrinsic order graph for $n = 1$.

Indeed I_1 contains a downward edge from 0 to 1 because (see Theorem 2.1) $0 \succ 1$, since matrix $\binom{0}{1}$ has no $\binom{1}{0}$ columns! Alternatively, using Theorem 2.2, we have that $0 \rhd 1$, since matrix $\binom{0}{1}$ has the pattern (3)! Moreover, this is in accordance with the obvious fact that

$$\Pr\{0\} = 1 - p_1 \geq p_1 = \Pr\{1\}, \text{ since } \quad p_1 \leq 1/2 \text{ due to (2)!}$$

However, for large values of n, a more efficient method is needed. For this purpose, in [4] the following algorithm for iteratively building up I_n (for all $n \geq 2$) from I_1 (depicted in Fig. 1), has been developed.

Theorem 2.3 (Building up I_n from I_1): Let $n \geq 2$. Then the graph of the poset $I_n = \{0, \ldots, 2^n - 1\}$ (on 2^n nodes) can be drawn simply by adding to the graph of the poset $I_{n-1} = \{0, \ldots, 2^{n-1} - 1\}$ (on 2^{n-1} nodes) its isomorphic copy $2^{n-1} + I_{n-1} = \{2^{n-1}, \ldots, 2^n - 1\}$ (on 2^{n-1} nodes). This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of I_n. Finally, the edges connecting one vertex u of I_{n-1} with the other vertex v of $2^{n-1} + I_{n-1}$ are given by the set of 2^{n-2} vertex pairs

$$\left\{ (u, v) \equiv \left(u_{(10}, 2^{n-2} + u_{(10} \right) \;\middle|\; 2^{n-2} \leq u_{(10} \leq 2^{n-1} - 1 \right\}.$$

Fig. 2 illustrates the above iterative process for the first few values of n, denoting all the binary n-tuples by their decimal equivalents. Basically, after adding to I_{n-1} its isomorphic copy $2^{n-1} + I_{n-1}$, we connect one-to-one the nodes of "the second half of the first half" to the nodes of "the first half of the second half": A nice fractal property of I_n!

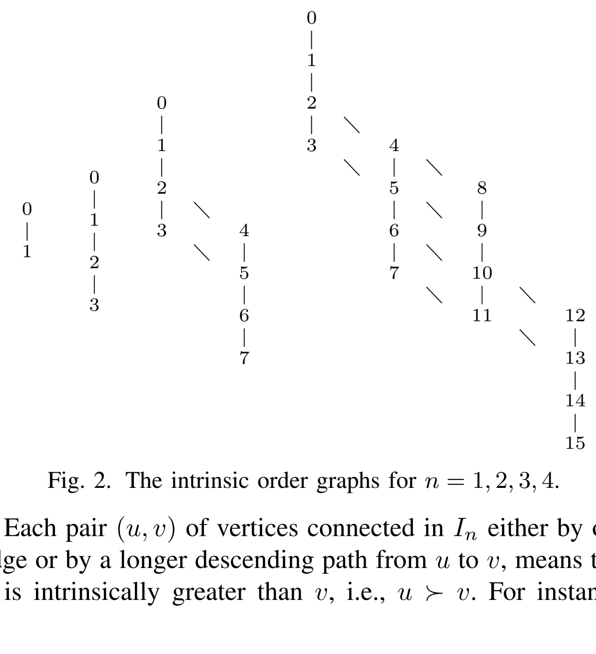

Fig. 2. The intrinsic order graphs for $n = 1, 2, 3, 4$.

Each pair (u, v) of vertices connected in I_n either by one edge or by a longer descending path from u to v, means that u is intrinsically greater than v, i.e., $u \succ v$. For instance, looking at the Hasse diagram of I_4, the right-most one in Fig. 2, we observe that $3 \equiv (0, 0, 1, 1) \succ 12 \equiv (1, 1, 0, 0)$, in accordance with Example 2.3.

On the contrary, each pair (u, v) of non-connected vertices in I_n either by one edge or by a longer descending path, means that u and v are incomparable by intrinsic order, i.e., $u \not\succ v$ and $v \not\succ u$. For instance, looking at the Hasse diagram of I_3, the third one from left to right in Fig. 2, we observe that $3 \equiv (0, 1, 1)$ and $4 \equiv (1, 0, 0)$ are incomparable by intrinsic order, in accordance with Example 2.2.

The edgeless graph for a given graph is obtained by removing all its edges, keeping its nodes at the same positions. In Fig. 3, the edgeless intrinsic order graph of I_5 is depicted.

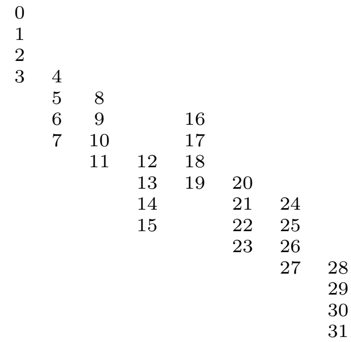

Fig. 3. The edgeless intrinsic order graph for $n = 5$.

For further theoretical properties and practical applications of the intrinsic order and the intrinsic order graph, we refer the reader to [5], [6], [7], [8], [9].

III. New Properties of the Intrinsic Order Graph

When viewed as an undirected graph, the Hasse diagram is called the cover graph of the poset. We refer the reader to [1], for standard notation and terminology concerning graphs. Using Theorems 2.1, 2.2, and 2.3 we can derive many different properties of the cover graph of I_n. Here, we have selected only a few of them.

A. Edges

Let V_n and E_n be the sets of vertices and edges, respectively, of I_n. As usual, $|A|$ denotes the cardinality of the set A. As mentioned, the number of nodes of I_n is obviously

$$|V_n| = |\{0, 1\}^n| = 2^n.$$

Our first property gives the number of edges of I_n.

Proposition 3.1: For all $n \geq 1$, the number of edges in the intrinsic order graph I_n is

$$|E_n| = (n + 1)\, 2^{n-2}. \tag{5}$$

Proof: The edges (going downward from u to v) in a Hasse diagram are exactly the covering relations ($u \rhd v$). Hence, using Theorem 2.2, we obtain

$$\begin{aligned}
|E_n| &= |\{(u, v) \in V_n \times V_n \mid u \rhd v\}| \\
&= |\{(u, v) \in V_n \times V_n \mid M_v^u \text{ has the pattern (3)}\}| + \\
&= |\{(u, v) \in V_n \times V_n \mid M_v^u \text{ has the pattern (4)}\}| \\
&= \left| \left\{ \begin{pmatrix} u_1 & \ldots & u_{n-1} & 0 \\ u_1 & \ldots & u_{n-1} & 1 \end{pmatrix} \right\} \right| + \\
&= \left| \left\{ \begin{pmatrix} u_1 & \ldots & u_{i-2} & 0 & 1 & u_{i+1} & \ldots & u_n \\ u_1 & \ldots & u_{i-2} & 1 & 0 & u_{i+1} & \ldots & u_n \end{pmatrix} \right\} \right| \\
&= 2^{n-1} + (n - 1)\, 2^{n-2} = (n + 1)\, 2^{n-2},
\end{aligned}$$

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as was to be shown. ■

Remark 3.1: Using proposition 3.1, we get for all $n \geq 2$

$$|E_n| = (n+1)\,2^{n-2} = 2 \cdot n \cdot 2^{n-3} + 2^{n-2} = 2\,|E_{n-1}| + 2^{n-2},$$

a recurrence relation for the number $|E_n|$ of edges of I_n, which could be also obtained directly from Theorem 2.2.

When we use the binary representation, the set E_n of all the $(n+1)\,2^{n-2}$ edges in I_n is given by Theorem 2.2. The following proposition gives this set using the decimal numbering for the pairs of adjacent nodes (see Fig. 2).

Proposition 3.2: For all $n \geq 1$

$$E_n = \left\{ \left(u_{(10}, u_{(10} + 1\right) \;\middle|\; \begin{array}{c} u_{(10} = 2p, \\ 0 \leq p \leq 2^{n-1} - 1 \end{array} \right\} \bigcup$$

$$\bigcup_{m=0}^{n-2} \left\{ \left(u_{(10}, u_{(10} + 2^m\right) \;\middle|\; \begin{array}{c} u_{(10} = q + 2^m\,(1+4r)\,, \\ 0 \leq q \leq 2^m - 1, \\ 0 \leq r \leq 2^{(n-2)-m} - 1 \end{array} \right\}.$$

Proof: The edges (going downward from u to v) in a Hasse diagram are exactly the covering relations ($u \rhd v$). So, using Theorem 2.2, we obtain

$$\begin{aligned} E_n &= \left\{ \left(u_{(10}, v_{(10}\right) \in V_n \times V_n \mid u \rhd v \right\} \\ &= \left\{ \left(u_{(10}, v_{(10}\right) \in V_n \times V_n \mid M_v^u \text{ has the pattern (3)} \right\} \\ &\cup \left\{ \left(u_{(10}, v_{(10}\right) \in V_n \times V_n \mid M_v^u \text{ has the pattern (4)} \right\}. \end{aligned}$$

On one hand, if M_v^u has the pattern (3) then we have that $v_{(10} = u_{(10} + 1$, and

$$\begin{aligned} u_{(10} &= (u_1, \ldots, u_{n-1}, 0)_{(10} \\ &= 2\,(u_1, \ldots, u_{n-1})_{(10} = 2p \;\left(0 \leq p \leq 2^{n-1} - 1\right). \end{aligned}$$

On the other hand, if M_v^u has the pattern (4) then making the change of variable $m = n - i$, we get

$$v_{(10} = u_{(10} + 2^{n-i} \text{ with } 2 \leq i \leq n, \text{ i.e.,}$$

$$v_{(10} = u_{(10} + 2^m \text{ with } 0 \leq m \leq n-2 \text{ and}$$

$$\begin{aligned} u_{(10} &= (u_1, \ldots, u_{i-2}, 0, 1, u_{i+1}, \ldots, u_n)_{(10} \\ &= (u_1, \ldots, u_{i-2}, 0, 0, 0, \ldots, 0)_{(10} \\ &+ (0, \ldots, 0, 0, 1, 0, \ldots, 0)_{(10} \\ &+ (0, \ldots, 0, 0, 0, u_{i+1}, \ldots, u_n)_{(10} \\ &= 2^{n-i+2}\,(u_1, \ldots, u_{i-2})_{(10} \\ &+ 2^{n-i} + (u_{i+1}, \ldots, u_n)_{(10} \\ &= 2^{m+2} r + 2^m + q = q + 2^m\,(1+4r)\,, \end{aligned}$$

where, $0 \leq q \leq 2^m - 1$ and $0 \leq r \leq 2^{(n-2)-m} - 1$. ■

Example 3.1: Let $n = 4$. Using Proposition 3.2, we get

$$\begin{aligned} A_4 &= \left\{ \left(u_{(10}, u_{(10} + 1\right) \;\middle|\; \begin{array}{c} u_{(10} = 2p, \\ 0 \leq p \leq 2^{n-1} - 1 = 7 \end{array} \right\} \\ &= \left\{ \begin{array}{c} (0,1)\,, (2,3)\,, (4,5)\,, (6,7)\,, \\ (8,9)\,, (10,11)\,, (12,13)\,, (14,15) \end{array} \right\}, \end{aligned}$$

$$\begin{aligned} B_4 &= \bigcup_{m=0}^{2} \left\{ \left(u_{(10}, u_{(10} + 2^m\right) \;\middle|\; \begin{array}{c} u_{(10} = q + 2^m\,(1+4r)\,, \\ 0 \leq q \leq 2^m - 1, \\ 0 \leq r \leq 2^{2-m} - 1 \end{array} \right\} \\ &= \left\{ \begin{array}{c} (1,2)\,, (5,6)\,, (9,10)\,, (13,14)\,, \\ (2,4)\,, (3,5)\,, (10,12)\,, (11,13)\,, \\ (4,8)\,, (5,9)\,, (6,10)\,, (7,11) \end{array} \right\}, \end{aligned}$$

where the three above rows respectively correspond to:

$$\begin{array}{llll} m = 0: & q = 0 & r = 0,1,2,3 & v_{(10} = u_{(10} + 2^0 \\ m = 1: & q = 0,1 & r = 0,1 & v_{(10} = u_{(10} + 2^1 \\ m = 2: & q = 0,1,2,3 & r = 0 & v_{(10} = u_{(10} + 2^2 \end{array}$$

Thus, $E_4 = A_4 \cup B_4$ contains all the 20 edges (pairs of adjacent nodes) of the graph I_4, as one can confirm looking at the right-most diagram in Fig. 2. Note that using (5) for $n = 4$, we can also confirm that the cardinality of E_4 is

$$|E_4| = (n+1)\,2^{n-2} = 5 \cdot 2^2 = 20.$$

B. Shadows, Neighbors and Degrees

The neighbors of a given vertex u in a graph, are all those nodes adjacent to u (i.e., connected by one edge to u). In particular, for (the cover graph of) a Hasse diagram, the neighbors of vertex u either cover u or are covered by u. This naturally leads to the following definition [10].

Definition 3.1: Let $(P, \leq)$ be a poset and $u \in P$. Then
(i) The lower shadow of u is the set

$$\Delta(u) = \{v \in P \mid v \text{ is covered by } u\} = \{v \in P \mid u \rhd v\}.$$

(ii) The upper shadow of u is the set

$$\nabla(u) = \{v \in P \mid v \text{ covers } u\} = \{v \in P \mid v \rhd u\}.$$

Particularly, for our poset $P = I_n$, regarding the lower shadow of $u \in \{0,1\}^n$, using Theorem 2.2, we have

$$\begin{aligned} \Delta(u) &= \{v \in \{0,1\}^n \mid u \rhd v\} \\ &= \{v \in \{0,1\}^n \mid M_v^u \text{ has the pattern (3)}\} \\ &\cup \{v \in \{0,1\}^n \mid M_v^u \text{ has the pattern (4)}\}, \end{aligned}$$

and hence, the cardinality of the lower shadow of u is exactly $1 - u_n$ (pattern (3)) plus the number of pairs of consecutive bits $(u_{i-1}, u_i) = (0,1)$ in u (pattern (4)). Formally:

$$|\Delta(u)| = (1 - u_n) + \sum_{i=2}^{n} \max\{u_i - u_{i-1}\,,\, 0\}. \quad (6)$$

Similarly, for the upper shadow of $u \in \{0,1\}^n$, using again Theorem 2.2, we have

$$\begin{aligned} \nabla(u) &= \{v \in \{0,1\}^n \mid v \rhd u\} \\ &= \{v \in \{0,1\}^n \mid M_u^v \text{ has the pattern (3)}\} \\ &\cup \{v \in \{0,1\}^n \mid M_u^v \text{ has the pattern (4)}\}, \end{aligned}$$

and hence, the cardinality of the upper shadow of u is exactly u_n (pattern (3)) plus the number of pairs of consecutive bits $(u_{i-1}, u_i) = (1,0)$ in u (pattern (4)). Formally:

$$|\nabla(u)| = u_n + \sum_{i=2}^{n} \max\{u_{i-1} - u_i\,,\, 0\}. \quad (7)$$

Next proposition provides the total number of neighbors of each node u of the intrinsic order graph I_n, the so-called degree of u, denoted, as usual, by $\delta(u)$.

Proposition 3.3: Let $n \geq 1$ and $u \in \{0,1\}^n$. The degree $\delta(u)$ of u (i.e., the number of neighbors of u) is

$$\delta(u) = 1 + \sum_{i=2}^{n} |u_i - u_{i-1}|. \quad (8)$$

Proof: Denoting by $N(u)$ the set of neighbors of a vertex $u \in \{0,1\}^n$ in the graph I_n, obviously we have

$$N(u) = \Delta(u) \cup \nabla(u)$$

and from (6) and (7), we immediately obtain

$$\delta(u) = |N(u)| = |\Delta(u)| + |\nabla(u)| = 1 + \sum_{i=2}^{n} |u_i - u_{i-1}|,$$

as was to be shown. ■

Next proposition provides us with the set of neighbors of each node u of the intrinsic order graph I_n, using decimal representation.

Proposition 3.4: Let $n \geq 1$, and let $u \in \{0,1\}^n$ with Hamming weight m. Write $u_{(10}$ as sum of powers of 2, in increasing order of the exponents, i.e.,

$$u_{(10} = \sum_{i=1}^{n} 2^{n-i} u_i = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_m} \tag{9}$$
$$(0 \leq p_1 < p_2 < \cdots < p_m \leq n-1).$$

(i) The lower shadow $\Delta(u)$ of u is characterized as follows:
(i)-(a) If $u_{(10}$ is even (i.e., if $u_n = 0$) then

$$u_{(10} + 1 \in \Delta(u), \text{ i.e., } u_{(10} \rhd u_{(10} + 1.$$

(i)-(b) For any power 2^p $(0 \leq p \leq n-2)$ in (9) s.t. 2^{p+1} does not appear in (9) then

$$u_{(10} + 2^p \in \Delta(u), \text{ i.e., } u_{(10} \rhd u_{(10} + 2^p.$$

(ii) The upper shadow $\nabla(u)$ of u is characterized as follows:
(ii)-(a) If $u_{(10}$ is odd (i.e., if $u_n = 1$) then

$$u_{(10} - 1 \in \nabla(u), \text{ i.e., } u_{(10} - 1 \rhd u_{(10}.$$

(ii)-(b) For any power 2^p $(1 \leq p \leq n-1)$ in (9) s.t. 2^{p-1} does not appear in (9) then

$$u_{(10} - 2^{p-1} \in \nabla(u), \text{ i.e., } u_{(10} - 2^{p-1} \rhd u_{(10}.$$

Proof: The assertions (i)-(a) and (ii)-(a) immediately follow using pattern (3) in Theorem 2.2, for matrices M_v^u and M_u^v, respectively. The assertions (i)-(b) and (ii)-(b) immediately follow using pattern (4) in Theorem 2.2, for matrices M_v^u and M_u^v, respectively. ■

Example 3.2: Let $n = 4$ and $u = (1,0,1,0)$. Then

$$u = (1,0,1,0) \equiv u_{(10} = 2^1 + 2^3 = 10.$$

Using Proposition 3.4-(i), we get (note that $u_{(10} = 10$ is even, i.e., $u_4 = 0$)

$$\Delta(10) = \{10+1\} \cup \{10+2^1\} = \{11, 12\}$$

and using Proposition 3.4-(ii), we get

$$\nabla(10) = \{10-2^0, 10-2^2\} = \{6, 9\}.$$

Thus (see the graph I_4, the right-most one in Fig. 2)

$$N(10) = \Delta(10) \cup \nabla(10) = \{6, 9, 11, 12\}$$

and using (8), we confirm that the cardinality of $N(10)$ is

$$\begin{aligned}\delta(10) = |N(10)| &= 1 + \sum_{i=2}^{4} |u_i - u_{i-1}| \\ &= 1 + |u_2 - u_1| + |u_3 - u_2| + |u_4 - u_3| \\ &= 1 + |0-1| + |1-0| + |0-1| = 4.\end{aligned}$$

C. Complementarity and Symmetry

Looking at any of the graphs in Figs. 2 & 3, we observe a "certain symmetry" in these diagrams. Let us formalize this fact.

Definition 3.2: Let $n \geq 1$ and $u \in \{0,1\}^n$.
(i) The complementary n-tuple of u is the n-tuple obtained by changing all its 0s into 1s and vice versa, i.e.,

$$(u_1, \ldots, u_n)^c = (1-u_1, \ldots, 1-u_n).$$

(ii) The complementary set of a subset $S \subseteq \{0,1\}^n$ is the set of complementary n-tuples of all the n-tuples of S, i.e.,

$$S^c = \{u^c \mid u \in S\}.$$

Remark 3.2: Note that for all $(u_1, \ldots, u_n) \in \{0,1\}^n$

$$(u_1, \ldots, u_n) + (u_1, \ldots, u_n)^c = (1, \ldots, 1) \equiv 2^n - 1.$$

Hence, the simplest way to verify that two binary n-tuples are complementary, when we use their decimal representations, is to check that they sum up to $2^n - 1$. For instance, the binary 3-tuples $2 \equiv (0,1,0)$ and $5 \equiv (1,0,1)$ are complementary, since $2+5 = 7 = 2^3 - 1$. Similarly, the complementary of the binary 4-tuple $4 \equiv (0,1,0,0)$ is $11 \equiv (1,0,1,1)$, since $(2^4 - 1) - 4 = 15 - 4 = 11$.

The reason underlying the symmetry of the intrinsic order graph is the duality property of the intrinsic order stated by the following proposition.

Proposition 3.5: Let $n \geq 1$ and $u, v \in \{0,1\}^n$. Then

$$u \rhd v \Leftrightarrow v^c \rhd u^c, \quad u \succeq v \Leftrightarrow v^c \succeq u^c.$$

Proof: Clearly, the $\binom{0}{0}$, $\binom{1}{1}$, $\binom{0}{1}$ and $\binom{1}{0}$ columns in matrix M_v^u, respectively become $\binom{1}{1}$, $\binom{0}{0}$, $\binom{0}{1}$ and $\binom{1}{0}$ columns in matrix $M_{u^c}^{v^c}$. Hence, using Theorem 2.2, we have that $u \rhd v$ iff M_v^u has either the pattern (3) or the pattern (4) iff $M_{u^c}^{v^c}$ respectively has either the pattern (3) or the pattern (4) iff $v^c \rhd u^c$. Finally, the right-hand equivalence immediately follows from the left-hand one and from the transitive property of the intrinsic order. ■

Many nice consequences can be derived from Proposition 3.5. Next corollary states only a few of them.

Corollary 3.1: Let $n \geq 1$. Let u and v be any two binary n-tuples placed at symmetric positions (with respect to the central point) in the graph I_n. Then
(i) u and v are complementary n-tuples, i.e., $v = u^c$, $u = v^c$.
(ii) The Hamming weights of u and v sum up to n.
(iii) $\Delta(u) = \nabla^c(u^c) = \nabla^c(v)$, $\nabla(u) = \Delta^c(u^c) = \Delta^c(v)$.
(iv) The sets of neighbors of u and v are complementary. In particular, u and v have the same degree.
(v) $u_{(10}$ is even (odd) $\Leftrightarrow$ $v_{(10}$ is odd (even).

Proof: (i) It is a direct consequence of Proposition 3.5.
(ii) It suffices to use (i) and the obvious fact that

$$w_H(u) + w_H(u^c) = n.$$

(iii) Using Definition 3.1, Proposition 3.5 and (i), we get:

$$\begin{aligned} w \in \Delta(u) &\Leftrightarrow u \rhd w \Leftrightarrow w^c \rhd u^c \Leftrightarrow w^c \in \nabla(u^c) \\ &\Leftrightarrow w \in \nabla^c(u^c) \Leftrightarrow w \in \nabla^c(v)\end{aligned}$$

and thus, taking complementaries, we get

$$\begin{aligned}\Delta(u) = \nabla^c(u^c) &\Rightarrow \Delta^c(u) = \nabla(u^c) \\ &\Rightarrow \nabla(u) = \Delta^c(u^c) = \Delta^c(v).\end{aligned}$$

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(iv) Using (iii), we get

$$N(u) = \Delta(u) \cup \nabla(u) = \nabla^c(v) \cup \Delta^c(v) \\ = [\nabla(v) \cup \Delta(v)]^c = N^c(v)$$

and consequently

$$\delta(u) = |N(u)| = |N^c(v)| = |N(v)| = \delta(v).$$

(v) Using (i), we get

$$u_{(10} \text{ is even (odd)} \Leftrightarrow u_n = 0\,(1) \\ \Leftrightarrow v_n = 1\,(0) \Leftrightarrow v_{(10} \text{ is odd (even)}$$

and this concludes the proof. ∎

Example 3.3: Let $n = 4$. The binary 4-tuples $u = 4$ and $v = 11$ are placed at symmetric positions (with respect to the central point) in the graph I_4 (see Fig. 2). Therefore:
(i) $4^c \equiv (0,1,0,0)^c = (1,0,1,1) \equiv 11 \; (4 + 11 = 2^4 - 1)$.
(ii) $w_H(0,1,0,0) + w_H(1,0,1,1) = 1 + 3 = 4 = n$.

$$(iii)\Delta(4) = \{5,8\}, \; \nabla(11) = \{7,10\}, \; \{5,8\}^c = \{7,10\}. \\ \nabla(4) = \{2\}, \quad \Delta(11) = \{13\}, \quad \{2\}^c = \{13\}.$$

(iv) $N(4) = \{2,5,8\}$, $N(11) = \{7,10,13\}$, $\{2,5,8\}^c = \{7,10,13\}$ and $\delta(4) = 3 = \delta(11)$.
(v) 4 is even, 11 is odd.

D. Isomorphic Subgraphs and Fractal Structure

A bisection of a graph is a partition of its vertex set into two (disjoint) subsets with half the vertices each [1]. The most natural way of bisecting the intrinsic order graph I_n is the following. The first and second half, respectively, of $\{0,1\}^n$ will be the subsets of binary n-tuples whose first component is $u_1 = 0$ and $u_1 = 1$, respectively. This procedure can be reiterated by successively bisecting, in the same way, each of the so-obtained subgraphs.

Let $n \geq 1$, $1 \leq k \leq n$ and let $\bar{u}_1, \ldots, \bar{u}_k \in \{0,1\}$ be k fixed binary digits. From now on, $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ denotes the subset of bitstrings of $\{0,1\}^n$ whose first or left-most k components are fixed, namely $u_1 = \bar{u}_1, \ldots, u_k = \bar{u}_k$; while its last or right-most $n-k$ components, $u_{k+1}, \ldots, u_n$, take all possible values (0 or 1). More precisely, $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ is the set

$$\left\{ (\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) \;\middle|\; (u_{k+1}, \ldots, u_n) \in \{0,1\}^{n-k} \right\}$$

and its cardinality is $|I_n^{\bar{u}_1,\ldots,\bar{u}_k}| = \left| \{0,1\}^{n-k} \right| = 2^{n-k}$.

Let us recall that two graphs $G(V,E)$ and $G^*(V^*,E^*)$ are said to be isomorphic if there exists an isomorphism of one of them to the other, i.e., an edge-preserving bijection [1]. That is, a graph isomorphism is a one-to-one mapping between the vertex sets $\Phi : V \to V^*$, which preserves adjacency, i.e., u, v are adjacent in G if and only if $\Phi(u), \Phi(v)$ are adjacent in G^*.

The self-similarity property or fractal structure that one can observe in Figs. 2 & 3, is an immediate consequence of the following proposition.

Proposition 3.6: Let $n \geq 1$ and $1 \leq k \leq n$. The 2^k equal-sized subgraphs $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ (each with 2^{n-k} nodes), obtained after k successive bisections of the intrinsic order graph I_n, are pair-wise isomorphic, and indeed all of them are isomorphic to the intrinsic order graph I_{n-k}.

Proof: Consider the following one-to-one mapping

$$\begin{array}{ccc} I_n^{\bar{u}_1,\ldots,\bar{u}_k} & \xrightarrow{\Phi} & I_{n-k} \\ (\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) & \longmapsto & (u_{k+1}, \ldots, u_n) \end{array}$$

Using Theorem 2.2, we have

$$(\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) \rhd (\bar{u}_1, \ldots, \bar{u}_k, v_{k+1}, \ldots, v_n)$$

if and only if

$$(u_{k+1}, \ldots, u_n) \rhd (v_{k+1}, \ldots, v_n),$$

so that Φ is an isomorphism of graphs, since it preserves the edges (covering relations). ∎

For instance, let $n = 5$ and $k = 3$. After $k = 3$ successive bisections of the intrinsic order graph I_5, the $2^k = 8$ subgraphs are the 8 isomorphic "columns" (each containing $2^{n-k} = 4$ nodes) depicted in Fig. 3. Moreover, any of these "column"-subgraphs of I_5 (5-tuples) is isomorphic to I_2 (2-tuples), the second graph from the left in Fig. 2.

IV. CONCLUSION

The behavior of a CSBS depends on the current values of the 2^n binary n-tuple probabilities and on the ordering between them. In this sense, the intrinsic order graph I_n provides us with an useful representation of a CSBS, by displaying all the bitstrings in decreasing order of their occurrence probabilities. In this paper, several new properties of the digraph I_n have been stated and rigorously proved (e.g., number of edges, neighbors and degrees of each vertex, symmetry, fractal structure, etc.). Each of these properties has been illustrated with a simple example and with the corresponding graph. Since many different technical systems in Reliability Engineering are indeed CSBSs, then our results can be applied to develop new (or to improve already known) algorithms –based on the intrinsic order– for evaluating the unavailability system. From a theoretical point of view, this paper suggests the search of new graph-theoretic and order-theoretic properties of the intrinsic order graph I_n.

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