

Desenredo y suavizado de mallas de tetraedros en el método del mecano

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Abstract

El *método del mecano* [13] es un procedimiento para generar mallas tridimensionales de un dominio definido por una triangulación de su superficie, Σ_s . La idea básica consiste en descomponer Σ_s en parches conexos, Σ_s^i , a los que se les asocian las caras externas de un *mecano* que aproxima el objeto y que se construye a partir de piezas elementales interconectadas. La parametrización de Floater establece una correspondencia biyectiva entre las caras exteriores del mecano y los parches Σ_s^i , de manera que un punto cualquiera de la superficie del mecano queda asociado a un único punto de Σ_s . En este trabajo nos centramos en el problema de cómo transformar los nodos interiores del mecano de manera que la malla tridimensional del objeto no contenga tetraedros invertidos y sea de buena calidad. Esta tarea se lleva a cabo mediante un proceso iterativo el que cada nodo de la malla se desplaza a una nueva posición que optimiza la *malla local*, esto es, el conjunto de tetraedros conectados al nodo *nodo libre*. En la optimización de la malla local utilizamos una variante de las funciones objetivo habituales que fue introducida en [4] y que, a diferencia de éstas, es capaz de actuar sobre mallas enredadas.

1 Introducción

The meccano technique proposed in [13] creates a 3-D triangulation of a solid defined by a triangulation of its surface. The meccano consists of a series of interconnected pieces

approximating the solid. These pieces are subsequently divided into tetrahedra by using the Kossaczky refinement [12]. The nodes of the triangulation of the meccano are mapped to the solid, resulting in a 3-D triangulation of the object. To this end, the surface of the solid Σ_s decomposes into connected patches, Σ_s^i , which are associated with the external faces of the meccano. Let us consider a graph in which each subtriangulation is a vertex of the graph and two vertices of the graph are connected if their corresponding subtriangulations have at least a common edge. Then, in order to have a proper association between patches and meccano faces, the graphs of the solid and the meccano must be identical. The Floater parametrization (see, for example [5]) establishes a one to one correspondence between the external faces of the meccano and the patches Σ_s^i , so that any point on the surface of the meccano has a unique image in Σ_s . Thus, any triangulation generated on the surface of the meccano will have as image a new triangulation of Σ_s . However, it is still necessary to determine how to transform the inner nodes of the meccano in order to get an admissible 3-D mesh of the solid, i.e. it does not contain inverted tetrahedra and has a good quality. In this paper we describe a smoothing and untangling procedure of tetrahedral meshes able to relocate the inner nodes, leading to a three-dimensional mesh of high quality.

The most usual techniques to improve the quality of a *valid* mesh, that is, one that does not have inverted elements, are based upon local smoothing. In short, these techniques consist of finding the new positions that the mesh nodes must hold, in such a way that they optimize an objective function. Such a function is based on a certain measurement of the quality of the *local submesh*, $N(v)$, formed by the set of tetrahedra connected to the *free node* v . As it is a local optimization process, we can not guarantee that the final mesh is globally optimum. Nevertheless, after repeating this process several times for all the nodes of the current mesh, quite satisfactory results can be achieved. Usually, objective functions are appropriate to improve the quality of a valid mesh, but they do not work properly when there are inverted elements. This is because they present singularities (barriers) when any tetrahedron of $N(v)$ changes the sign of its Jacobian determinant. To avoid this problem Freitag et al proposed a procedure where the optimization is carry out in two stages. In the first one, the possible inverted elements are untangled by an algorithm that maximizes their negative Jacobian determinants [8]; in the second, the resulting mesh from the first stage is smoothed using another objective function based on a quality metric of the tetrahedra of $N(v)$ [9]. One of these objective functions are present in Section 2. After the untangling procedure, the mesh has a very poor quality because the technique has no motivation to create good-quality elements. As remarked in [6], it is not possible to apply a gradient-based algorithm to optimize the objective function because it is not continuous all over \mathbb{R}^3 , making it necessary to use other non-standard approaches.

We propose an alternative to this procedure, such that the untangling and smoothing are carried out in the same stage. For this purpose, we use a suitable modification of the objective function such that it is regular all over \mathbb{R}^3 . When a feasible region (subset of \mathbb{R}^3 where v could be placed, being $N(v)$ a valid submesh) exists, the minima of the original and modified objective functions are very close and, when this region does not exist, the minimum of the modified objective function is located in such a way that it tends to untangle $N(v)$. The latter occurs, for example, when the fixed boundary of $N(v)$ is tangled. With this approach, we can use any standard and efficient unconstrained optimization method to find the minimum of the modified objective function, see for

example [2].

In this work we have applied the proposed modification to one objective function derived from an *algebraic mesh quality metric* studied in [10], but it would also be possible to apply it to other objective functions which have barriers like those presented in [11]. The results for two test problems are shown in Section 4. Finally, conclusions and future research are presented in Section 5.

2 Objective Functions

Several *tetrahedron shape measures* [3] could be used to construct an objective function. Nevertheless those obtained by algebraic operations are specially indicated for our purpose because they can be computed very efficiently. The above mentioned algebraic mesh quality metric and the corresponding objective function are shown in this Section.

Let T be a tetrahedral element in the physical space whose vertices are given by $\mathbf{x}_k = (x_k, y_k, z_k)^T \in \mathbb{R}^3$, $k = 0, 1, 2, 3$ and T_R be the reference tetrahedron with vertices $\mathbf{u}_0 = (0, 0, 0)^T$, $\mathbf{u}_1 = (1, 0, 0)^T$, $\mathbf{u}_2 = (0, 1, 0)^T$ and $\mathbf{u}_3 = (0, 0, 1)^T$. If we choose \mathbf{x}_0 as the translation vector, the affine map that takes T_R to T is $\mathbf{x} = A\mathbf{u} + \mathbf{x}_0$, where A is the Jacobian matrix of the affine map referenced to node \mathbf{x}_0 , and expressed as $A = (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \mathbf{x}_3 - \mathbf{x}_0)$.

Let now T_I be an equilateral tetrahedron with all its edges of length one and vertices located at $\mathbf{v}_0 = (0, 0, 0)^T$, $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (1/2, \sqrt{3}/2, 0)^T$, $\mathbf{v}_3 = (1/2, \sqrt{3}/6, \sqrt{2}/\sqrt{3})^T$. Let $\mathbf{v} = W\mathbf{u}$ be the linear map that takes T_R to T_I , being $W = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ its Jacobian matrix.

Therefore, the affine map that takes T_I to T is given by $\mathbf{x} = AW^{-1}\mathbf{v} + \mathbf{x}_0$, and its Jacobian matrix is $S = AW^{-1}$. This weighted matrix S is independent of the node chosen as reference; it is said to be *node invariant* [10]. We can use matrix norms, determinant or trace of S to construct algebraic quality measures of T . For example, the Frobenius norm of S , defined by $|S| = \sqrt{\text{tr}(S^T S)}$, is specially indicated because it is easily computable. Thus, it is shown in [10] that $q = \frac{3\sigma^{\frac{2}{3}}}{|S|^2}$ is an algebraic quality measure of T , where $\sigma = \det(S)$. The maximum value of these quality measures is the unity and it corresponds to equilateral tetrahedron. Besides, any flat tetrahedron has quality measure zero. We can derive an optimization function from this quality measure. Thus, let $\mathbf{x} = (x, y, z)^T$ be the free node position of v , and let S_m be the weighted Jacobian matrix of the m -th tetrahedron of $N(v)$. We define the objective function of \mathbf{x} , associated to an m -th tetrahedron as

$$\eta_m = \frac{|S_m|^2}{3\sigma_m^{\frac{2}{3}}} \quad (1)$$

Then, the corresponding objective function for $N(v)$ can be constructed by using the p -norm of $(\eta_1, \eta_2, \dots, \eta_M)$ as

$$|K_\eta|_p(\mathbf{x}) = \left[\sum_{m=1}^M \eta_m^p(\mathbf{x}) \right]^{\frac{1}{p}} \quad (2)$$

where M is the number of tetrahedra in $N(v)$. The objective function $|K_\eta|_1$ was deduced and used in [1] for smoothing and adapting of 2-D meshes. Finally, this function, among

others, is studied and compared in [11]. We note that the cited authors only use this objective function for smoothing valid meshes.

Although this optimization function is smooth in those points where $N(v)$ is a valid submesh, it becomes discontinuous when the volume of any tetrahedron of $N(v)$ goes to zero. It is due to the fact that η_m approaches infinity when σ_m tends to zero and its numerator is bounded below. In fact, it is possible to prove that $|S_m|$ reaches its minimum, with strictly positive value, when v is placed in the geometric center of the fixed face of the m -th tetrahedron. The positions where v must be located to get $N(v)$ to be valid, i.e., the feasible region, is the interior of the polyhedral set P defined as $P = \bigcap_{m=1}^M H_m$, where H_m are the half-spaces defined by $\sigma_m(\mathbf{x}) \geq 0$. This set can occasionally be empty, for example, when the fixed boundary of $N(v)$ is tangled. In this situation, function $|K_\eta|_p$ stops being useful as optimization function. On the other hand, when the feasible region exists, that is $\text{int } P \neq \emptyset$, the objective function tends to infinity as v approaches the boundary of P . Due to these singularities, a barrier is formed which avoids reaching the appropriate minimum by using gradient-based algorithms, when these start from a free node outside the feasible region. In other words, with these algorithms we can not optimize a tangled mesh $N(v)$ with the above objective function.

3 Modified Objective Functions

We propose a modification in the previous objective function (2), so that the barrier associated with its singularities will be eliminated and the new function will be smooth all over \mathbb{R}^3 . An essential requirement is that the minima of the original and modified functions are nearly identical when $\text{int } P \neq \emptyset$. Our modification consists of substituting σ in (2) by the positive and increasing function

$$h(\sigma) = \frac{1}{2}(\sigma + \sqrt{\sigma^2 + 4\delta^2}) \quad (3)$$

being the parameter $\delta = h(0)$. We represent in Fig. 1 the function $h(\sigma)$. Thus, the new objective function here proposed is given by

$$|K_\eta^*|_p(\mathbf{x}) = \left[\sum_{m=1}^M (\eta_m^*)^p(\mathbf{x}) \right]^{\frac{1}{p}} \quad (4)$$

where

$$\eta_m^* = \frac{|S_m|^2}{3h^{\frac{2}{3}}(\sigma_m)} \quad (5)$$

is the modified objective function for the m -th tetrahedron.

The behavior of $h(\sigma)$ in function of δ parameter is such that, $\lim_{\delta \rightarrow 0} h(\sigma) = \sigma$, $\forall \sigma \geq 0$ and $\lim_{\delta \rightarrow 0} h(\sigma) = 0$, $\forall \sigma \leq 0$. Thus, if $\text{int } P \neq \emptyset$, then $\forall \mathbf{x} \in \text{int } P$ we have $\sigma_m(\mathbf{x}) > 0$, for $m = 1, 2, \dots, M$ and, as smaller values of δ are chosen, $h(\sigma_m)$ behaves very much as σ_m , so that, the original objective function and its corresponding modified version are very close in the feasible region. Particularly, in the feasible region, as $\delta \rightarrow 0$, function $|K_\eta^*|_p$ converges

pointwise to $|K_\eta|_p$. Besides, by considering that $\forall \sigma > 0$, $\lim_{\delta \rightarrow 0} h'(\sigma) = 1$ and $\lim_{\delta \rightarrow 0} h^{(n)}(\sigma) = 0$, for $n \geq 2$, it is easy to prove that the derivatives of this objective function verify the same property of convergence. As a result of these considerations, it may be concluded that the positions of v that minimize original and modified objective functions are nearly identical when δ is *small*. Actually, the value of δ is selected in terms of point v under consideration, making it as small as possible and in such a way that the evaluation of the minimum of modified functions does not present any computational problem. Suppose that $\text{int } P = \emptyset$,

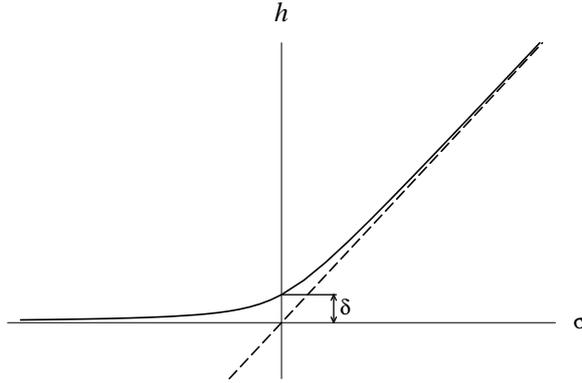


Figure 1: Representation of function $h(\sigma)$.

then the original objective function, $|K_\eta|_p$, is not suitable for our purpose because it is not correctly defined. Nevertheless, modified function is well defined and tends to solve the tangle. We can reason it from a qualitative point of view by considering that the dominant terms in $|K_\eta^*|_p$ are those associated to the tetrahedra with more negative values of σ and, therefore, the minimization of these terms imply the increase of these values. It must be remarked that $h(\sigma)$ is an increasing function and $|K_\eta^*|_p$ tends to ∞ when the volume of any tetrahedron of $N(v)$ tends to $-\infty$, since $\lim_{\sigma \rightarrow -\infty} h(\sigma) = 0$.

In conclusion, by using the modified objective function, we can untangle the mesh and, at the same time, improve its quality. Obviously, the modification here proposed can be easily applied to other objective functions.

For a better understanding of the behavior of the objective function and its modification, we propose the following 2-D test example. Let us consider a simple 2-D mesh formed by three triangles, vBC , vCA and vAB , where we have fixed $A(0, -1)$, $B(\sqrt{3}, 0)$, $C(0, 1)$ and $v(x, y)$ is the free node. In this case, the feasible region is the interior of the equilateral triangle ABC . In Fig. 2(a) we show $|K_\eta|_2$ (solid line) and $|K_\eta^*|_2$ (dashed line) as a function of x for a fixed value $y = 0$ (the y -coordinate of the optimal solution). The chosen parameter δ is 0.1. We can see that original objective function presents several local minima and discontinuities, opposite to the modified one. Besides, the original function reach their absolute minimum outside the feasible region. Vertical asymptotes in original objective function correspond to positions of the free node for which $\sigma = 0$ for any tetrahedra of the local mesh. As might be expected, the optimal solution for the modified function results in $v(\sqrt{3}/3, 0)$. The original and modified functions are nearly identical in the proximity of this point, see Fig. 2(a).

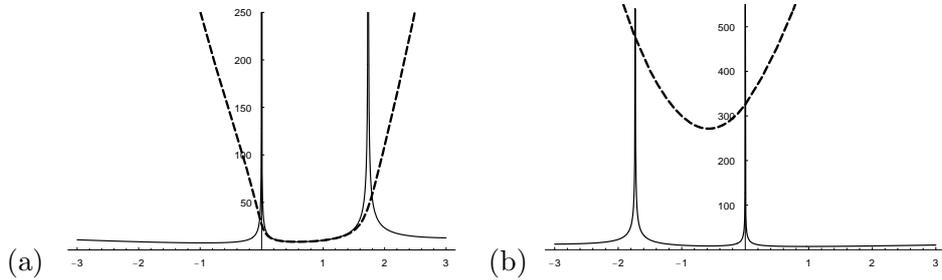


Figure 2: (a) Transversal cut of $|K_\eta|_2$ (solid line) and $|K_\eta^*|_2$ (dashed line) for the 2-D test example; (b) the same objective functions for the tangled mesh.

Let us now consider the tangled mesh obtained by changing the position of point $B(\sqrt{3}, 0)$ to $B'(-\sqrt{3}, 0)$. Here, the mesh is constituted by the triangles $vB'C$, vCA and vAB' , where $vB'C$ and vAB' are inverted. The feasible region does not exist in this new situation. The graphics of functions $|K_\eta|_2$ and $|K_\eta^*|_2$ are represented in Fig. 2(b). Although the mesh can not be untangled, we get $v(-\sqrt{3}/3, 0)$ as the optimal position of the free node by using our modified objective function. For this position the three triangles are “equally inverted” (same negative values of σ). In this example the same result could be achieved by maximizing the minimum value of σ in the mesh, as proposed in [8].

4 Applications

To check the efficiency of the proposed techniques we first consider a regular mesh of a unit cube with 750 tetrahedra, 216 nodes uniformly distributed and a maximum valence of 16. In order to get a tangled test mesh, we transform the unit cube into a greater one ($10 \times 10 \times 10$) changing the positions of some nodes and preserving their connectivities. The inner nodes remains in their original positions, the nodes sited on the edges of the unit cube are replaced on the edges of new cube and, finally, the interior nodes of each face of the unit cube are projected on the corresponding face of the new cube. The initial tangled mesh, shown in Fig. 3(a), has 10 inverted tetrahedra and an average quality measure of $q_{avg} = 0.384$ (the average quality of the regular mesh is 0.749). Besides, approximately the 50% of tetrahedra has a very poor quality (less than 0.04). Here we have chosen the quality measure proposed in [6], $q = \frac{3}{|S_m||S_m^{-1}|}$, for valid tetrahedra and $q = 0$ for inverted ones. The result after twenty four sweeps of the mesh optimization process with $|K_\eta^*|_2$ is shown in Fig. 3(b). In this case, the steepest descent algorithm was used for the optimization of the objective function. In Fig. 4 we present the evolution of the average quality measure, q_{avg} , and the minimal quality, q_{min} , in terms of the number of iterations of the mesh optimization process. Note that the average quality initially decreases because the number of inverted tetrahedra increases in former iterations. The mesh has 22 inverted tetrahedra after the first iteration, 33 after the second, 16 after the third, 11 after the fourth and 0 after the fifth.

We also present an application of the Armadillo’s figure, remeshed with the meccano technique. The final mesh has 54496 tetrahedra and 13015 nodes. We have used a cube,

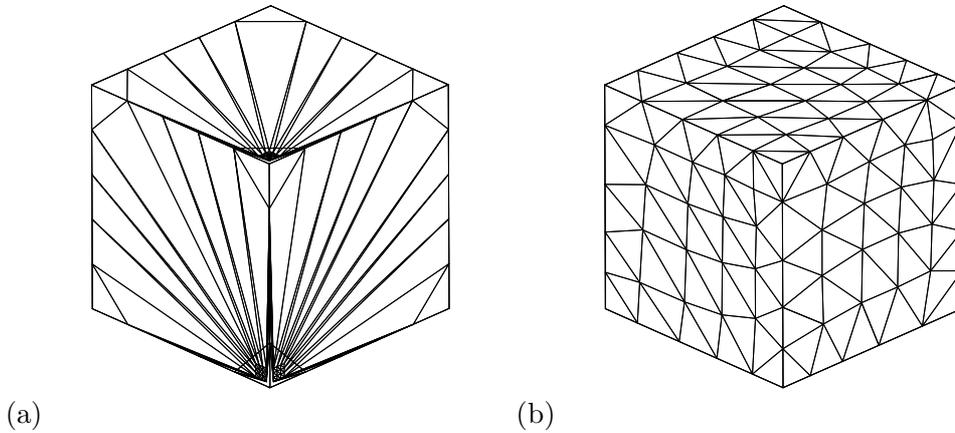


Figure 3: (a) Initial tangled mesh of a cube and (b) the resulting mesh after twenty four steps of the optimization process.

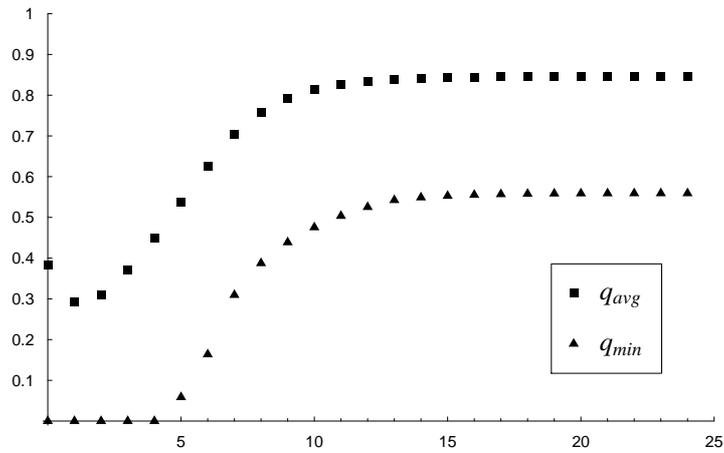


Figure 4: Values of the average quality q_{avg} and the minimal quality q_{min} in terms of the number of iterations of the mesh optimization process for the cube test.

sited close to the Armadillo's back, as a parametric space. In Fig. 5 (a) it is shown the resulting mesh after the nodes, initially sited on the faces of the cube, have been mapped to the true surface. Note that there are many tangled tetrahedra (2384) crossing the surface because the remainder nodes stay inside the cube. A transversal cut of this mesh is shown in Fig. 5 (b). The same cut, after applying the untangling and smoothing procedure, is shown in Fig. 5 (c). Finally, the optimized mesh of Armadillo is shown in Fig. 5 (d). It has an average quality of 0.68 and only has one tetrahedron with a quality less than 0.1.

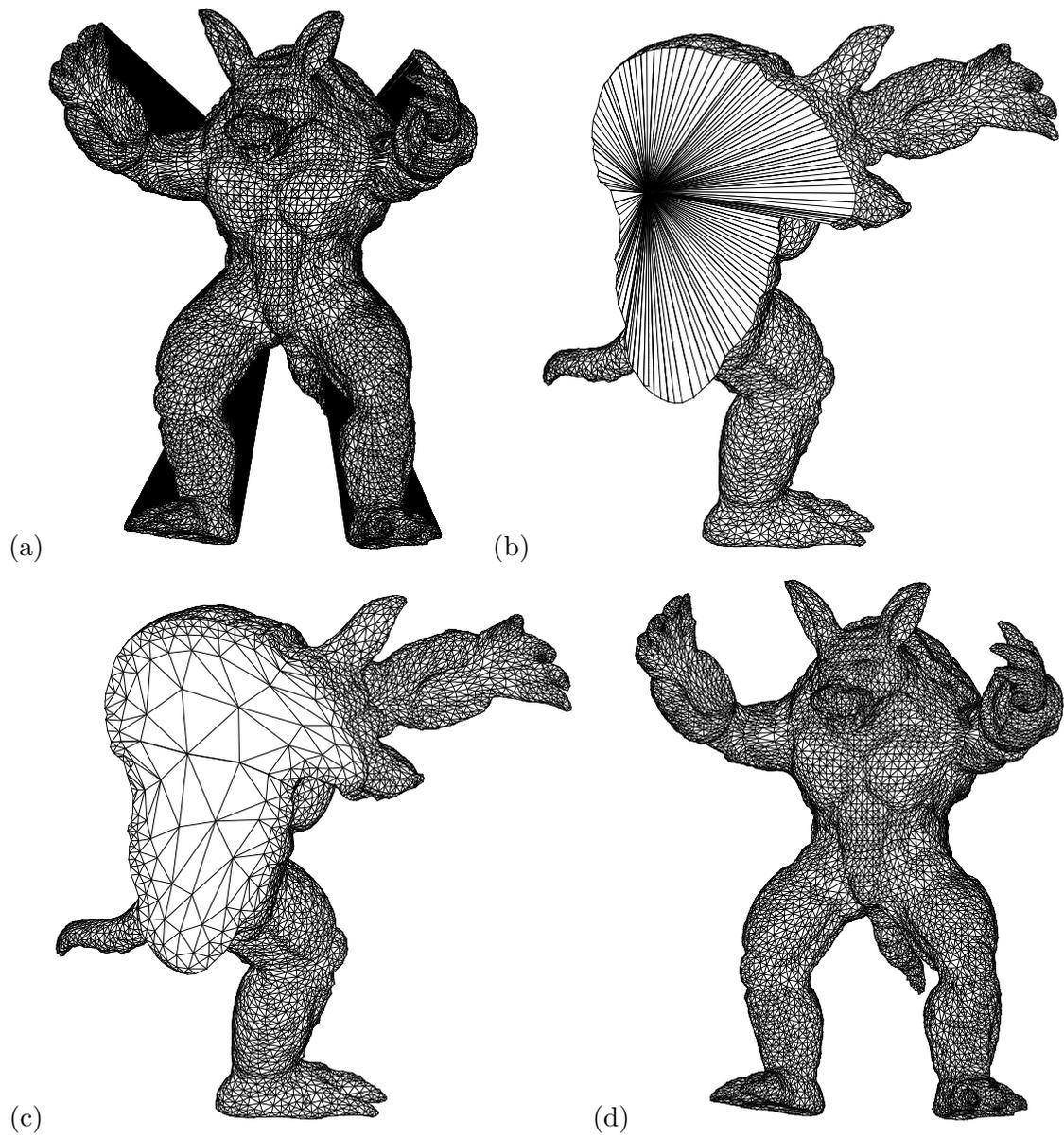


Figure 5: Armadillo remeshed with the meccano technique

5 Conclusions

In this paper we present a way to avoid the singularities of common objective functions used to optimize tetrahedral meshes. To do so, we propose a modification of these functions in such a way that it makes them regular all over \mathbb{R}^3 . Thus, the modified objective functions can be used to smooth and untangle the mesh simultaneously. The regularity shown by the modified objective functions allows the use of standard optimization algorithms as steepest descent, conjugate gradient, quasi-Newton, etc.

The smoothing and untangling procedure is the key to relocate the inner nodes of the meshes constructed by the mecano method. We have proved with numerous examples that this optimized meshes have a high quality.

These techniques can be implemented in a parallel algorithm, as reported in [7], in order to reduce the computational time of the process.

Bibliography

- [1] R.E. Bank, R.K.Smith, *Mesh Smoothing Using a Posteriori Error Estimates*. SIAM J. Numer. Anal. **34** (1997), 979–997
- [2] M.S. Bazaraa, H.D. Sherali, C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, John Wiley and Sons, Inc., New York, 1993
- [3] J. Dompierre, P. Labbé, F. Guibault, R. Camarero, *Proposal of Benchmarks for 3D Unstructured Tetrahedral Mesh Optimization*. In Proc. 7th International Meshing Roundtable. Sandia National Laboratories (1998), 459–478
- [4] J.M. Escobar, E. Rodríguez, R. Montenegro, G. Montero, J.M. González-Yuste, *Simultaneous untangling and smoothing of tetrahedral meshes*. J. Comput. Meth. Appl. Mech. Engng., 192 25 (2003), 2775–2787.
- [5] M.S. Floater, L.A. Freitag, M. Jones, P. Plassmann, *Mean Value Coordinates*, Comput. Aid. Geom. Design, 20 (2003) 19–27.
- [6] L.A. Freitag, P.M. Knupp, *Tetrahedral Element Shape Optimization Via the Jacobian Determinant and Condition Number*. In Proc. of the 8th International Meshing Roundtable. Sandia National Laboratories (1999), 247–258
- [7] L.A. Freitag, M. Jones, P. Plassmann, *A Parallel Algorithm for Mesh Smoothing*. SIAM J. Sci. Comput. **20** (2000), 2023–2040
- [8] L.A. Freitag, P. Plassmann, *Local Optimization-based Simplicial Mesh Untangling and Improvement*. Int. J. Numer. Meth. Eng. **49** (2000), 109–125
- [9] L.A. Freitag, P.M. Knupp, *Tetrahedral Mesh Improvement via Optimization of the Element Condition Number*. Int. J. Numer. Meth. Eng. **53** (2002), 1377–1391
- [10] P.M. Knupp, *Algebraic Mesh Quality Metrics*. SIAM J. Sci. Comput. **23** (2001), 193–218
- [11] P.M. Knupp, *Achieving Finite Element Mesh Quality via Optimization of the Jacobian Matrix Norm and Associated Quantities. Part II - A Frame Work for Volume Mesh Optimization and the Condition Number of the Jacobian Matrix*. Int. J. Numer. Meth. Eng. **48** (2000), 1165–1185
- [12] I. Kossaczky, *A recursive approach to local mesh refinement in two and three dimensions*. J. Comput. Appl. Math. 55, (1994.) 275–288
- [13] R. Montenegro, J.M. Cascón, J.M. Escobar, E. Rodríguez, G. Montero *An automatic strategy for adaptive tetrahedral mesh generation*. Applied Numerical Mathematics, 2009. DOI: 10.1016/j.apnum.2008.12.010.