A Multilayered Convection Diffusion Model

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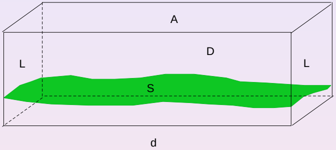
1. PRESENTATION
   - Convection Diffusion Model
   - Change of coordinates
2. Numerical Method
   - A Finite Element - Characteristic - Finite Difference method
3. Numerical Results

Convection Diffusion Model



$$\frac{\partial u}{\partial t} + V.\nabla_{xy} u + W\frac{\partial u}{\partial z} - \nabla(k_{xy}\nabla_{xy} u) - \frac{\partial}{\partial z}(k_z \frac{\partial u}{\partial z}) = f \quad in\Omega \quad (1)$$

$$-k_{xy}\nabla u.\nu|_\Gamma = [V.\nu]^+ u \quad (2)$$

$$u\big|_{t=0} = 0 \quad (3)$$

Change of coordinates

We make a change of coordinates in order to transform the domain into a cuboid.

$$\begin{aligned} \tau &= t \\ \xi &= x \\ \eta &= y \\ \varsigma &= z - h(x,y) \end{aligned}$$

The equations in the transformed domain are as in (1),(2),(3) replacing W by $W - V_1 \frac{\partial h}{\partial x} - V_2 \frac{\partial h}{\partial y}$
and k_z by $\left(k_\zeta + k_{\xi\eta}(\frac{\partial h}{\partial x})^2 + k_{\xi\eta}(\frac{\partial h}{\partial y})^2\right)$
plus terms with croosed derivatives in (1)

A Finite Element - Characteristic - Finite Difference method

For $l = 1, ..., L$

$$\frac{u_l^{n+1/4} - \bar{u}_l^n}{\Delta t/2} = 0$$

$$\begin{aligned}
\frac{u_l^{n+1/2} - u_l^{n+1/4}}{\Delta t} &+ \frac{W_l^+}{\Delta z}(u_l^{n+1/2} - u_{l-1}^{n+1/2}) \\
&+ \frac{1}{2} A_l u_l^{n+1/2} + k_z \frac{-u_{l-1}^{n+1/2} + u_l^{n+1/2}}{(\Delta z)^2} \\
&\{if(l == 1) + \lambda u^{n+1/2}\} = \frac{1}{2} f^{n+1/2}
\end{aligned}$$

A Finite Element - Characteristic - Finite Difference method

For $l = L, ..., 1$

$$\frac{u_l^{n+3/4} - u_l^{n+1/2}}{\Delta t} - \frac{W_l^-}{\Delta z}(u_l^{n+3/4} - u_{l-1}^{n+3/4}) + \frac{1}{2}A_l u_l^{n+3/4} + k_z \frac{u_l^{n+3/4} - u_{l+1}^{n+3/4}}{(\Delta z)^2} \{if(l == 1) + \lambda u^{n+3/4}\} = \frac{1}{2} f^{n+3/4}$$

$$\frac{u_l^{n+1} - \bar{u}_l^{n+3/4}}{\Delta t/2} = 0$$

Numerical Examples

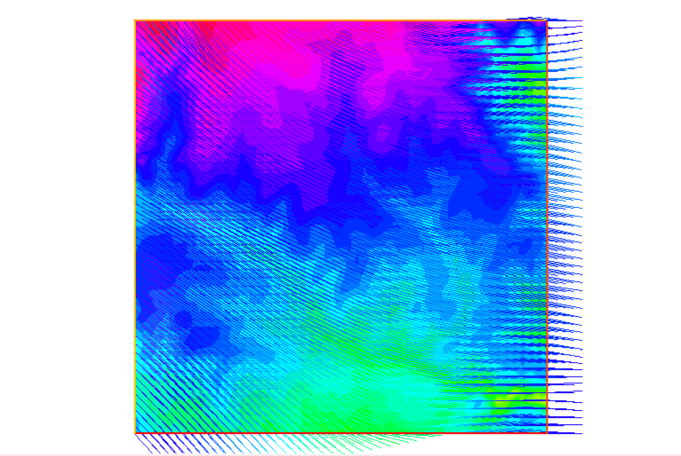


Figure: velocity field

$$f(t,x) = ae^{-(\frac{\log(2)}{c}t)}e^{-((X[0]-x[0])^2+(X[1]-x[1])^2)/(2b^2))}$$

- t is the time in seconds
- $a = 100$ pre-exponential factor.
- $b = 100$ is the standar deviation of the gaussian distribution
- $c = 300$ is the half life time of the emision in seconds.
- $X = [500, 4500]^t$ is the point where the emision takes place.

The other physical values are

- Horizontal Diffusion $k_{xy} = 10^{-1}$
- Vertical Diffusion $k_z = 10^{-3}$
- Absortion coefficient in the surface level $\lambda = 0.001$

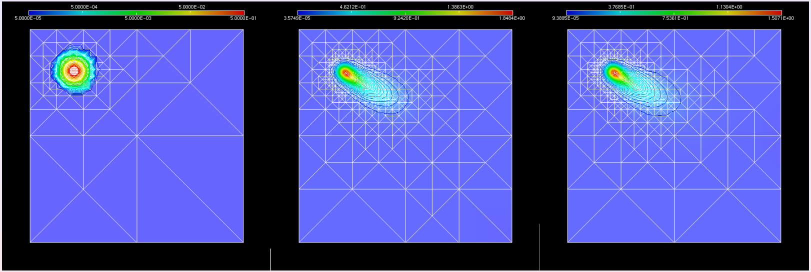


Figure: Concentration in the first level at different time steps

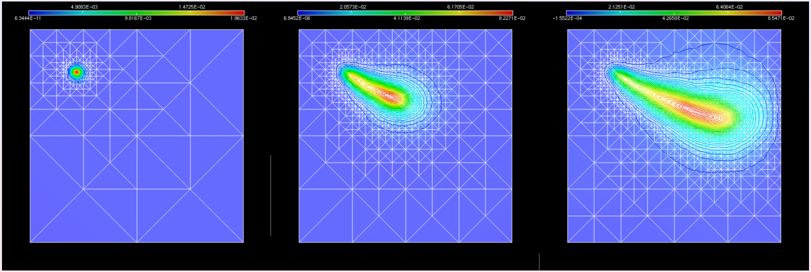


Figure: Concentration in the third level at different time steps

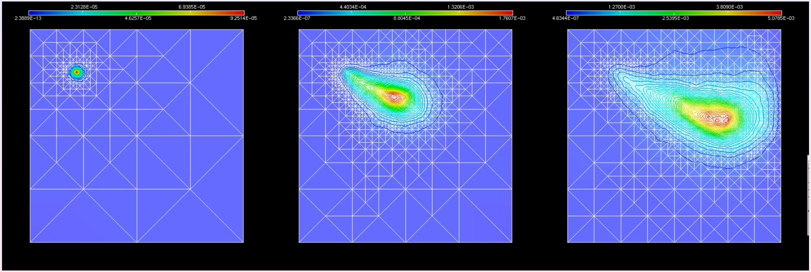


Figure: Concentration in the fifth level at different time steps