Edges, Chains, Shadows, Neighbors and Subgraphs in the Intrinsic Order Graph

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***Abstract*—Many different scientific, technical or social phenomena can be modeled by a complex system depending on a large number n of random Boolean variables. Such systems are called complex stochastic Boolean systems (CSBSs). The most useful representation of a CSBS is the intrinsic order graph. This is a symmetric digraph on 2^n nodes, with a characteristic fractal structure. In this paper, different properties of the intrinsic order graph are studied, namely those dealing with its edges; chains; shadows, neighbors and degrees of its vertices; and some relevant subgraphs, as well as the natural isomorphisms between them.**

***Index Terms*—complex stochastic Boolean system, edges, intrinsic order graph, neighbors, shadows, subgraphs.**

I. Introduction

In this paper, we consider complex systems depending on an arbitrary number n of random Boolean variables $x_1, \ldots, x_n$, the so-called *complex stochastic Boolean systems* (CSBSs). That is, the n system basic components x_i are assumed to be stochastic (i.e., non-deterministic), and they only take two possible values: 0, 1.

So, each one of the 2^n possible situations (outcomes) for a CSBS is given by a binary n-tuple $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ of 0s and 1s, and it has its own occurrence probability $\Pr\{(u_1, \ldots, u_n)\}$ [11]. Throughout this paper, the n basic components of the system are assumed to be statistically independent.

Using the classical terminology in Statistics, a stochastic Boolean system can be modeled by the n-dimensional Bernoulli distribution $X = (x_1, \ldots, x_n)$ with sample space $\{0,1\}^n$, and parameters $p_1, \ldots, p_n$ defined by

$$\Pr\{x_i = 1\} = p_i, \ \Pr\{x_i = 0\} = 1 - p_i,$$

so that, taking into account the statistical independence of the Bernoulli marginal variables x_i, for all $u \in \{0,1\}^n$, we have

$$\Pr\{\, u\} = \prod_{i=1}^{n} \Pr\{x_i = u_i\} = \prod_{i=1}^{n} p_i^{u_i} \left(1 - p_i\right)^{1-u_i}, \quad (1)$$

that is, $\Pr\{\, u\}$ is the product of factors p_i if $u_i = 1$, $1 - p_i$ if $u_i = 0$.

Example 1.1: Let $n = 4$ and $u = (1,0,1,0) \in \{0,1\}^4$. Let $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.3$, $p_4 = 0.4$. Then using (1), we have

$$\Pr\{(1,0,1,0)\} = p_1 (1 - p_2)\, p_3 (1 - p_4) = 0.0144.$$

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The behavior of a CSBS is determined by the ordering between the current values of the 2^n associated binary n-tuple probabilities $\Pr\{u\}$. Computing all these 2^n probabilities –by using (1)– and ordering them in decreasing or increasing order of their values is only possible in practice for small values of the number n of basic variables. However, for large values of n, to overcome the exponential nature of this problem, we need alternative procedures for comparing the binary string probabilities. For this purpose, in [2] we have defined a partial order relation on the set $\{0,1\}^n$ of all the 2^n binary n-tuples, the so-called *intrinsic order* between binary n-tuples.

The intrinsic order relation is characterized by a simple positional criterion, the so-called *intrinsic order criterion* (IOC). IOC enables one to compare (to order) two given binary n-tuple probabilities $\Pr\{u\}, \Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the binary n-tuples u, v.

More precisely, for those pairs (u, v) of binary n-tuples comparable by intrinsic order, the ordering between their occurrence probabilities is always the same for all sets of basic probabilities $\{p_i\}_{i=1}^n$. On the contrary, for those pairs (u, v) of binary n-tuples incomparable by intrinsic order, the ordering between their occurrence probabilities depends on the current values the set of basic probabilities $\{p_i\}_{i=1}^n$.

The most useful graphical representation of a CSBS is the intrinsic order graph. This is a symmetric, self-dual diagram on 2^n nodes (denoted by I_n) that displays all the binary n-tuples from top to bottom in decreasing order of their occurrence probabilities. Formally, I_n is the Hasse diagram of the intrinsic partial order relation on $\{0,1\}^n$.

In this context, the main goal of this paper is to present some new properties of the intrinsic order graph. In particular, we give the set and the number of edges of I_n, the set and the number of elements which are neighbors (adjacent) in the graph to a fixed binary n-tuple $u \in \{0,1\}^n$. To determine the set of neighbors of a given binary n-tuple u, we first study its lower and upper shadows. Moreover, we also analyze some chains and subgraphs of the the intrinsic order graph. Some of these properties can be found in [9], but this paper also presents some other new properties of I_n, not described in that paper.

For this purpose, we have organized this paper as follows. In Section II, we present some notations, definitions, and previous results about the intrinsic order and the intrinsic order graph, in order to make this paper self-contained. Section III is devoted to present some properties of the intrinsic order graph, concerning its edges and chains. In Section IV, the lower and upper shadows and the set of neighbors of an arbitrary node are studied. In Section V, some special subgraphs of I_n are analyzed. Finally, in Section VI, we present our conclusions.

II. The Intrinsic Order

Throughout this paper, we indistinctly denote the n-tuple $u \in \{0,1\}^n$ by its binary representation $(u_1,\ldots,u_n)$ or by its decimal representation, and we use the symbol "$\equiv$" to indicate the conversion between these two numbering systems. The decimal numbering and the Hamming weight (i.e., the number of 1-bits) of u will be respectively denoted by

$$u \equiv u_{(10} = \sum_{i=1}^{n} 2^{n-i} u_i, \quad w_H(u) = \sum_{i=1}^{n} u_i.$$

Example 2.1: Let $n=6$ and $u=(1,0,1,0,1,1)$. Then

$$u = (1,0,1,0,1,1) \equiv 2^0 + 2^1 + 2^3 + 2^5 = 43,$$

$$w_H(u) = 4.$$

Given two binary n-tuples $u,v \in \{0,1\}^n$, the ordering between their occurrence probabilities $\Pr(u)$, $\Pr(v)$ obviously depends on the Bernoulli parameters p_i, as the following simple example shows.

Example 2.2: Let $n=3$, $u=(0,1,1)$ and $v=(1,0,0)$. For $p_1=0.1,\ p_2=0.2,\ p_3=0.3$, using (1), we have:

$$\Pr\{(0,1,1)\} = 0.054 < \Pr\{(1,0,0)\} = 0.056,$$

for $p_1=0.2,\ p_2=0.3,\ p_3=0.4$, using (1), we have:

$$\Pr\{(0,1,1)\} = 0.096 > \Pr\{(1,0,0)\} = 0.084.$$

However, as mentioned in Section I, in [2] we have established an intrinsic, positional criterion to compare the occurrence probabilities of two given binary n-tuples without computing them. This criterion is presented in detail in Section II-*A*, while its graphical representation is shown in Section II-*B*.

A. The Intrinsic Order Relation

Theorem 2.1 (The intrinsic order theorem): Let $n \geq 1$. Let $x_1,\ldots,x_n$ be n mutually independent Bernoulli variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy

$$0 < p_1 \leq p_2 \leq \cdots \leq p_n \leq 0.5. \tag{2}$$

Then the occurrence probability of the binary n-tuple v, i.e., $v=(v_1,\ldots,v_n) \in \{0,1\}^n$, is *intrinsically* less than or equal to the occurrence probability of the binary n-tuple u, i.e., $u=(u_1,\ldots,u_n) \in \{0,1\}^n$, (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (2)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 2.1: In the following, we assume that the parameters p_i always satisfy condition (2). Fortunately, this hypothesis is not restrictive for practical applications.

Remark 2.2: The $\binom{0}{1}$ column preceding each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

Remark 2.3: The term *corresponding*, used in Theorem 2.1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding each other. In other words, for each $\binom{1}{0}$ column in matrix M_v^u the number of preceding $\binom{0}{1}$ columns must be strictly greater than the number of preceding $\binom{1}{0}$ columns.

Claim 2.1: IOC can be equivalently reformulated in the following way, involving only the 1-bits of u and v (with no need to use their 0-bits). Matrix M_v^u satisfies IOC if and only if either u has no 1-bits (i.e., u is the zero n-tuple) or for each 1-bit in u there exists (at least) one corresponding 1-bit in v placed at the same or at a previous position. In other words, either u has no 1-bits or for each 1-bit in u, say $u_i=1$, the number of 1-bits in $(v_1,\ldots,v_i)$ must be greater than or equal to the number of 1-bits in $(u_1,\ldots,u_i)$.

The matrix condition IOC, stated by Theorem 2.1 or by Claim 2.1, is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it only depends on the relative positions of the 0s and 1s in the binary strings u and v. Theorem 2.1 naturally leads to the following partial order relation on the set $\{0,1\}^n$ [2], [3]. The so-called intrinsic order will be denoted by "$\preceq$", and when $v \preceq u$ we say that v *is intrinsically less than or equal to* u (or u *is intrinsically greater than or equal to* v).

Definition 2.1: For all $u,v \in \{0,1\}^n$

$$v \preceq u \ \text{ iff } \ \Pr\{v\} \leq \Pr\{u\} \ \text{ for all set } \{p_i\}_{i=1}^n \text{ s.t. (2)}$$

$$\text{iff matrix } M_v^u \text{ satisfies IOC.}$$

In the following, the partially ordered set (poset, for short) for n variables $(\{0,1\}^n, \preceq)$ will be denoted by I_n; see [12] for more details about posets.

Example 2.3: For $n=3$:

$$3 \equiv (0,1,1) \npreceq (1,0,0) \equiv 4 \ \&\ (1,0,0) \npreceq (0,1,1) \ \text{ since}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

do not satisfy IOC (Remark 2.3). Therefore, $(0,1,1)$ and $(1,0,0)$ are incomparable by intrinsic order, i.e., the ordering between $\Pr\{(0,1,1)\}$ and $\Pr\{(1,0,0)\}$ depends on the basic probabilities p_i, as Example 2.2 has shown.

Example 2.4: For $n=4$:

$$12 \equiv (1,1,0,0) \preceq (0,1,0,1) \equiv 5 \ \text{ since}$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

satisfies IOC (Remark 2.2). For all $0 < p_1 \leq \cdots \leq p_4 \leq \frac{1}{2}$

$$\Pr\{(1,1,0,0)\} \leq \Pr\{(0,1,0,1)\}.$$

Example 2.5: For all $n \geq 1$, the binary n-tuples

$$\left(0, \overset{n}{\ldots}, 0\right) \equiv 0 \quad \text{and} \quad \left(1, \overset{n}{\ldots}, 1\right) \equiv 2^n - 1$$

are the maximum and minimum elements, respectively, in the poset I_n. Indeed, both matrices

$$\begin{pmatrix} 0 & \ldots & 0 \\ u_1 & \ldots & u_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1 & \ldots & u_n \\ 1 & \ldots & 1 \end{pmatrix}$$

satisfy the intrinsic order criterion, since they have no $\binom{1}{0}$ columns!.

Thus, for all $u \in \{0,1\}^n$ and for all $\{p_i\}_{i=1}^n$ s.t. (2)

$$\Pr\left\{\left(1, \overset{n}{\ldots}, 1\right)\right\} \leq \Pr\{(u_1,\ldots,u_n)\} \leq \Pr\left\{\left(0, \overset{n}{\ldots}, 0\right)\right\}.$$

Many different properties of the intrinsic order can be immediately derived from its simple matrix description IOC [2], [3], [5]. For instance, we have the two following necessary (but not sufficient) conditions for intrinsic order (see [3] for the proof).

Corollary 2.1: For all $u, v \in \{0,1\}^n$

$$u \succeq v \Rightarrow w_H(u) \le w_H(v),$$
$$u \succeq v \Rightarrow u_{(10} \le v_{(10}.$$

B. The Intrinsic Order Graph

In this subsection, the graphical representation of the poset $I_n = (\{0,1\}^n, \preceq)$ is presented. The usual representation of a poset is its Hasse diagram (see [12] for more details about these diagrams). Specifically, for our poset I_n, its Hasse diagram is a directed graph (digraph, for short) whose vertices are the 2^n binary n-tuples of 0s and 1s, and whose edges go upward from v to u whenever u covers v, denoted by $u \rhd v$. This means that u is intrinsically greater than v with no other elements between them, i.e.,

$$u \rhd v \quad \Leftrightarrow \quad u \succ v \text{ and } \nexists \; w \in \{0,1\}^n \text{ s.t. } u \succ w \succ v.$$

A simple matrix characterization of the covering relation for the intrinsic order is given in the next theorem; see [4] for the proof.

Theorem 2.2 (Covering relation in I_n): Let $n \ge 1$ and $u, v \in \{0,1\}^n$. Then $u \rhd v$ if and only if the only columns of matrix M_v^u different from $\binom{0}{0}$ and $\binom{1}{1}$ are either its last column $\binom{0}{1}$ or just two columns, namely one $\binom{1}{0}$ column immediately preceded by one $\binom{0}{1}$ column, i.e., either

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{n-1} & 0 \\ u_1 & \dots & u_{n-1} & 1 \end{pmatrix} \text{ or} \tag{3}$$

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{i-2} & 0 & 1 & u_{i+1} & \dots & u_n \\ u_1 & \dots & u_{i-2} & 1 & 0 & u_{i+1} & \dots & u_n \end{pmatrix}. \tag{4}$$
$$(2 \le i \le n)$$

Example 2.6: For $n = 4$, we have

$$6 \rhd 7 \text{ since } M_7^6 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \text{ has the pattern (3),}$$

$$10 \rhd 12 \text{ since } M_{12}^{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ has the pattern (4).}$$

The Hasse diagram of the poset I_n will be also called the *intrinsic order graph* for n variables, denoted as well by I_n.

For small values of n, the intrinsic order graph I_n can be directly constructed by using either Theorem 2.1 or Theorem 2.2. For instance, for $n = 1$: $I_1 = (\{0,1\}, \preceq)$, and its Hasse diagram is shown in Fig. 1.

Fig. 1. The intrinsic order graph for $n = 1$.

Indeed I_1 contains a downward edge from 0 to 1 because (see Theorem 2.1) $0 \succ 1$, since matrix $\binom{0}{1}$ has no $\binom{1}{0}$ columns! Alternatively, using Theorem 2.2, we have that $0 \rhd 1$, since matrix $\binom{0}{1}$ has the pattern (3)! Moreover, this is in accordance with the obvious fact that

$$\Pr\{0\} = 1 - p_1 \ge p_1 = \Pr\{1\}, \text{ since } \; p_1 \le 1/2 \text{ due to (2)!}$$

However, for large values of n, a more efficient method is needed. For this purpose, in [4] the following algorithm for iteratively building up I_n (for all $n \ge 2$) from I_1 (depicted in Fig. 1), has been developed.

Theorem 2.3 (Building up I_n from I_1): Let $n \ge 2$. Then the graph of the poset $I_n = \{0, \dots, 2^n - 1\}$ (on 2^n nodes) can be drawn simply by adding to the graph of the poset $I_{n-1} = \{0, \dots, 2^{n-1} - 1\}$ (on 2^{n-1} nodes) its isomorphic copy $2^{n-1} + I_{n-1} = \{2^{n-1}, \dots, 2^n - 1\}$ (on 2^{n-1} nodes). This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of I_n. Finally, the edges connecting one vertex u of I_{n-1} with the other vertex v of $2^{n-1} + I_{n-1}$ are given by the set of 2^{n-2} vertex pairs

$$\left\{(u,v) \equiv \left(u_{(10}, 2^{n-2} + u_{(10}\right) \;\middle|\; 2^{n-2} \le u_{(10} \le 2^{n-1} - 1\right\}.$$

Fig. 2 illustrates the above iterative process for the first few values of n, denoting all the binary n-tuples by their decimal equivalents. Basically, after adding to I_{n-1} its isomorphic copy $2^{n-1} + I_{n-1}$, we connect one-to-one the nodes of "the second half of the first half" to the nodes of "the first half of the second half": A nice fractal property of I_n!

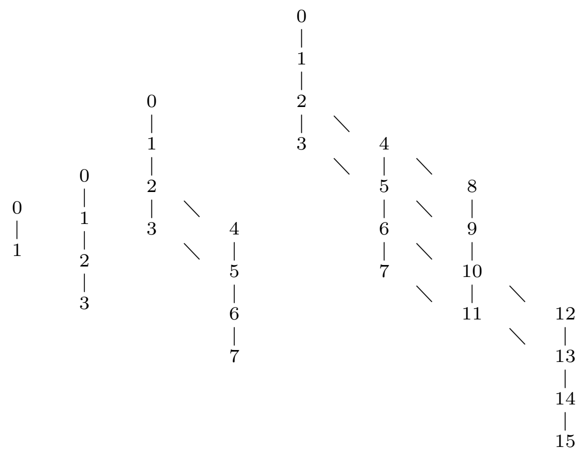

Fig. 2. The intrinsic order graphs for $n = 1, 2, 3, 4$.

Each pair (u, v) of vertices connected in I_n either by one edge or by a longer descending path from u to v, means that u is intrinsically greater than v, i.e., $u \succ v$. For instance, looking at the Hasse diagram of I_4, the right-most one in Fig. 2, we observe that $5 \equiv (0,1,0,1) \succ 12 \equiv (1,1,0,0)$, in accordance with Example 2.4.

On the contrary, each pair (u, v) of non-connected vertices in I_n either by one edge or by a longer descending path, means that u and v are incomparable by intrinsic order, i.e., $u \not\succ v$ and $v \not\succ u$. For instance, looking at the Hasse diagram of I_3, the third one from left to right in Fig. 2, we observe that $3 \equiv (0,1,1)$ and $4 \equiv (1,0,0)$ are incomparable by intrinsic order, in accordance with Example 2.3.

Moreover, the properties of the intrinsic order stated by Example 2.5 and Corollary 2.1, are also illustrated by any of the diagrams in Fig. 2.

The edgeless graph for a given graph is obtained by removing all its edges, keeping its nodes at the same positions. In Fig. 3, the edgeless intrinsic order graph of I_5 is depicted.

0							
1							
2							
3	4						
	5	8					
	6	9		16			
	7	10		17			
		11	12	18			
			13	19	20		
			14		21	24	
			15		22	25	
					23	26	
						27	28
							29
							30
							31

Fig. 3. The edgeless intrinsic order graph for $n = 5$.

For further theoretical properties and practical applications of the intrinsic order and the intrinsic order graph, we refer the reader to [5], [6], [7], [8], [9], [10].

When viewed as the natural representation of a partial order relation, the Hasse diagram of the intrinsic order is just the picture of the poset I_n. We refer the reader to [12], for more details about posets. When viewed as an undirected graph, the Hasse diagram is called the cover graph of the poset. We refer the reader to [1], for standard notation and terminology concerning graphs. Using Theorems 2.1, 2.2, and 2.3 we can derive many different order-theoretic and graph-theoretic properties of I_n. In Sections III, IV, and V, some of these properties are presented.

III. Edges, Chains and Chain Decompositions in the Intrinsic Order Graph

A. Edges

Let V_n and E_n be the sets of vertices and edges, respectively, of I_n. As usual, $|A|$ denotes the cardinality of the set A. As mentioned, the number of nodes of I_n is obviously

$$|V_n| = |\{0,1\}^n| = 2^n.$$

Our first property gives the number of edges of I_n.

Proposition 3.1: For all $n \geq 1$, the number of edges in the intrinsic order graph I_n is

$$|E_n| = (n+1)\,2^{n-2}. \tag{5}$$

Proof: The edges (going downward from u to v) in a Hasse diagram are exactly the covering relations ($u \rhd v$). Hence, using Theorem 2.2, we obtain

$$\begin{aligned}
|E_n| &= |\{(u,v) \in V_n \times V_n \mid u \rhd v\}| \\
&= |\{(u,v) \in V_n \times V_n \mid M_v^u \text{ has the pattern (3)}\}| + \\
&= |\{(u,v) \in V_n \times V_n \mid M_v^u \text{ has the pattern (4)}\}| \\
&= \left|\left\{\begin{pmatrix} u_1 & \ldots & u_{n-1} & 0 \\ u_1 & \ldots & u_{n-1} & 1 \end{pmatrix}\right\}\right| + \\
&= \left|\left\{\begin{pmatrix} u_1 & \ldots & u_{i-2} & 0 & 1 & u_{i+1} & \ldots & u_n \\ u_1 & \ldots & u_{i-2} & 1 & 0 & u_{i+1} & \ldots & u_n \end{pmatrix}\right\}\right| \\
&= 2^{n-1} + (n-1)\,2^{n-2} = (n+1)\,2^{n-2},
\end{aligned}$$

as was to be shown. ■

Remark 3.1: Using proposition 3.1, we get for all $n \geq 2$

$$|E_n| = (n+1)\,2^{n-2} = 2{\cdot}n{\cdot}2^{n-3} + 2^{n-2} = 2\,|E_{n-1}| + 2^{n-2},$$

a recurrence relation for the number $|E_n|$ of edges of I_n, which could be also obtained directly from Theorem 2.2.

When we use the binary representation, the set E_n of all the $(n+1)\,2^{n-2}$ edges in I_n is given by Theorem 2.2. The following proposition gives this set using the decimal numbering for the pairs of adjacent nodes (see Fig. 2).

Proposition 3.2: For all $n \geq 1$

$$E_n = \left\{ \left(u_{(10}, u_{(10} + 1\right) \;\middle|\; \begin{array}{c} u_{(10} = 2p, \\ 0 \leq p \leq 2^{n-1} - 1 \end{array} \right\} \bigcup$$

$$\bigcup_{m=0}^{n-2} \left\{ \left(u_{(10}, u_{(10} + 2^m\right) \;\middle|\; \begin{array}{c} u_{(10} = q + 2^m\,(1+4r)\,, \\ 0 \leq q \leq 2^m - 1, \\ 0 \leq r \leq 2^{(n-2)-m} - 1 \end{array} \right\}.$$

Proof: The edges (going downward from u to v) in a Hasse diagram are exactly the covering relations ($u \rhd v$). So, using Theorem 2.2, we obtain

$$\begin{aligned}
E_n &= \left\{ \left(u_{(10}, v_{(10}\right) \in V_n \times V_n \mid u \rhd v \right\} \\
&= \left\{ \left(u_{(10}, v_{(10}\right) \in V_n \times V_n \mid M_v^u \text{ has the pattern (3)} \right\} \\
&\cup \left\{ \left(u_{(10}, v_{(10}\right) \in V_n \times V_n \mid M_v^u \text{ has the pattern (4)} \right\}.
\end{aligned}$$

On one hand, if M_v^u has the pattern (3) then we have that $v_{(10} = u_{(10} + 1$, and

$$\begin{aligned}
u_{(10} &= (u_1, \ldots, u_{n-1}, 0)_{(10} \\
&= 2\,(u_1, \ldots, u_{n-1})_{(10} = 2p \;\left(0 \leq p \leq 2^{n-1} - 1\right).
\end{aligned}$$

On the other hand, if M_v^u has the pattern (4) then making the change of variable $m = n - i$, we get

$$v_{(10} = u_{(10} + 2^{n-i} \text{ with } 2 \leq i \leq n, \text{ i.e.,}$$

$$v_{(10} = u_{(10} + 2^m \text{ with } 0 \leq m \leq n-2 \text{ and}$$

$$\begin{aligned}
u_{(10} &= (u_1, \ldots, u_{i-2}, 0, 1, u_{i+1}, \ldots, u_n)_{(10} \\
&= (u_1, \ldots, u_{i-2}, 0, 0, 0, \ldots, 0)_{(10} \\
&+ (0, \ldots, 0, 0, 1, 0, \ldots, 0)_{(10} \\
&+ (0, \ldots, 0, 0, 0, u_{i+1}, \ldots, u_n)_{(10} \\
&= 2^{n-i+2}\,(u_1, \ldots, u_{i-2})_{(10} \\
&+ 2^{n-i} + (u_{i+1}, \ldots, u_n)_{(10} \\
&= 2^{m+2}r + 2^m + q = q + 2^m\,(1+4r)\,,
\end{aligned}$$

where, $0 \leq q \leq 2^m - 1$ and $0 \leq r \leq 2^{(n-2)-m} - 1$. ■

Example 3.1: Let $n = 4$. Using Proposition 3.2, we get

$$\begin{aligned}
A_4 &= \left\{ \left(u_{(10}, u_{(10} + 1\right) \;\middle|\; \begin{array}{c} u_{(10} = 2p, \\ 0 \leq p \leq 2^{n-1} - 1 = 7 \end{array} \right\} \\
&= \left\{ \begin{array}{c} (0,1)\,, (2,3)\,, (4,5)\,, (6,7)\,, \\ (8,9)\,, (10,11)\,, (12,13)\,, (14,15) \end{array} \right\},
\end{aligned}$$

$$\begin{aligned}
B_4 &= \bigcup_{m=0}^{2} \left\{ \left(u_{(10}, u_{(10} + 2^m\right) \;\middle|\; \begin{array}{c} u_{(10} = q + 2^m\,(1+4r)\,, \\ 0 \leq q \leq 2^m - 1, \\ 0 \leq r \leq 2^{2-m} - 1 \end{array} \right\} \\
&= \left\{ \begin{array}{c} (1,2)\,, (5,6)\,, (9,10)\,, (13,14)\,, \\ (2,4)\,, (3,5)\,, (10,12)\,, (11,13)\,, \\ (4,8)\,, (5,9)\,, (6,10)\,, (7,11) \end{array} \right\},
\end{aligned}$$

where the three above rows respectively correspond to:

$$\begin{array}{llll}
m = 0: & q = 0 & r = 0,1,2,3 & v_{(10} = u_{(10} + 2^0 \\
m = 1: & q = 0,1 & r = 0,1 & v_{(10} = u_{(10} + 2^1 \\
m = 2: & q = 0,1,2,3 & r = 0 & v_{(10} = u_{(10} + 2^2
\end{array}$$

Thus, $E_4 = A_4 \cup B_4$ contains all the 20 edges (pairs of adjacent nodes) of the graph I_4, as one can confirm looking at the right-most diagram in Fig. 2. Note that using (5) for $n = 4$, we can also confirm that the cardinality of E_4 is

$$|E_4| = (n+1)\,2^{n-2} = 5 \cdot 2^2 = 20.$$

B. Chains

Two elements u, v of a poset $(P, \leq)$ are said to be comparable if either $u \leq v$ or $v \leq u$. A chain in a poset is a totally ordered subset, i.e., a subset of pairwise comparable elements. A chain $u = u^1 > u^2 > \cdots > u^l = v$ from u to v is said to have length $l-1$. A chain is said to be saturated when no further elements can be interpolated between its elements. In other words, all successive relations in a saturated chain $u^1 > u^2 > \cdots > u^l$ are coverings [12].

In particular, a saturated chain of length $l-1$ in our poset I_n is a subset $\{u^1, u^2, \ldots, u^l\}$ of $\{0,1\}^n$, such that $u^1 \rhd u^2 \rhd \cdots \rhd u^l$, i.e., $u^1 \succ u^2 \succ \cdots \succ u^l$ with no other elements between them.

A chain decomposition of a poset P is a family of disjoint chains whose union is P. A chain cover of a poset P is a chain decomposition into saturated chains, i.e., a set of disjoint saturated chains covering the elements of P.

Let us mention that one can define many different chain covers of I_n. The chain cover of our poset consisting of the largest possible number of chains (namely, 2^{n-1}), with the smallest possible length (namely, 1) is stated in the following Proposition. Basically, the idea is the following: Each even number $2k$ covers its consecutive odd number $2k+1$.

Proposition 3.3: For all $n \geq 1$ the poset I_n can be partitioned into the following 2^{n-1} saturated chains of length 1, that we call "congruence chains $(\bmod 2)$":

$$2k \rhd 2k+1 \quad \left(0 \leq k \leq 2^{n-1}-1\right).$$

Proof: For all $k \equiv (u_1, \ldots, u_{n-1}) \in \{0,1\}^{n-1}$, matrix

$$M_{2k+1}^{2k} = \begin{pmatrix} u_1 & \ldots & u_{n-1} & 0 \\ u_1 & \ldots & u_{n-1} & 1 \end{pmatrix}$$

has the pattern (3). Finally, since all these chains are pairwise disjoint, and they completely cover I_n, i.e.,

$$\bigcup_{0 \leq k \leq 2^{n-1}-1} \{2k, 2k+1\} = [0, 2^n - 1] \equiv \{0,1\}^n,$$

the proof is concluded. ■

However, the most intuitive or natural way for partitioning I_n into saturated chains is clearly suggested by Figs. 2 or 3. Just consider the 2^{n-2} "columns" obtained after $n-2$ successive bisections of I_n, containing four consecutive numbers, and beginning with a multiple $4k$ of 4. More precisely

Proposition 3.4: For all $n \geq 2$ the poset I_n can be partitioned into the following 2^{n-2} saturated chains of length 3, that we call "congruence chains $(\bmod 4)$":

$$4k \rhd 4k+1 \rhd 4k+2 \rhd 4k+3 \quad \left(0 \leq k \leq 2^{n-2}-1\right).$$

Proof: For all $k \equiv (u_1, \ldots, u_{n-2}) \in \{0,1\}^{n-2}$, the matrices

$$M_{4k+1}^{4k} = \begin{pmatrix} u_1 & \ldots & u_{n-2} & 0 & 0 \\ u_1 & \ldots & u_{n-2} & 0 & 1 \end{pmatrix},$$

$$M_{4k+2}^{4k+1} = \begin{pmatrix} u_1 & \ldots & u_{n-2} & 0 & 1 \\ u_1 & \ldots & u_{n-2} & 1 & 0 \end{pmatrix},$$

$$M_{4k+3}^{4k+2} = \begin{pmatrix} u_1 & \ldots & u_{n-2} & 1 & 0 \\ u_1 & \ldots & u_{n-2} & 1 & 1 \end{pmatrix}$$

have either the pattern (3) or the pattern (4). Finally, since all these chains are pairwise disjoint, and they completely cover I_n, i.e.,

$$\bigcup_{0 \leq k \leq 2^{n-2}-1} \{4k, 4k+1, 4k+2, 4k+3\} = [0, 2^n - 1] \equiv \{0,1\}^n,$$

the proof is concluded. ■

For instance, for $n = 5$ the $2^{n-2} = 8$ "columns" or congruence chains $(\bmod 4)$ of the graph I_5 (depicted in Fig. 3), are shown in Fig. 4.

0	4	8	12	16	20	24	28
1	5	9	13	17	21	25	29
2	6	10	14	18	22	26	30
3	7	11	15	19	23	27	31

Fig. 4. The chain cover into saturated congruence chains $(\bmod 4)$ of the poset I_5.

IV. Shadows, Neighbors and Degrees in the Intrinsic Order Graph

A. Shadows

The following definition (see [12]) deals with the general theory of posets.

Definition 4.1: Let $(P, \leq)$ be a poset and $u \in P$. Then
(i) The lower shadow of u is the set

$$\Delta(u) = \{v \in P \mid v \text{ is covered by } u\} = \{v \in P \mid u \rhd v\}.$$

(ii) The upper shadow of u is the set

$$\nabla(u) = \{v \in P \mid v \text{ covers } u\} = \{v \in P \mid v \rhd u\}.$$

Particularly, for our poset $P = I_n$, regarding the lower shadow of $u \in \{0,1\}^n$, using Theorem 2.2, we have

$$\begin{aligned} \Delta(u) &= \{v \in \{0,1\}^n \mid u \rhd v\} \\ &= \{v \in \{0,1\}^n \mid M_v^u \text{ has the pattern (3)}\} \\ &\cup \{v \in \{0,1\}^n \mid M_v^u \text{ has the pattern (4)}\}, \end{aligned}$$

and hence, the cardinality of the lower shadow of u is exactly $1 - u_n$ (pattern (3)) plus the number of pairs of consecutive bits $(u_{i-1}, u_i) = (0,1)$ in u (pattern (4)). Formally:

$$|\Delta(u)| = (1 - u_n) + \sum_{i=2}^{n} \max\{u_i - u_{i-1}, 0\}. \quad (6)$$

Similarly, for the upper shadow of $u \in \{0,1\}^n$, using again Theorem 2.2, we have

$$\begin{aligned} \nabla(u) &= \{v \in \{0,1\}^n \mid v \rhd u\} \\ &= \{v \in \{0,1\}^n \mid M_u^v \text{ has the pattern (3)}\} \\ &\cup \{v \in \{0,1\}^n \mid M_u^v \text{ has the pattern (4)}\}, \end{aligned}$$

and hence, the cardinality of the upper shadow of u is exactly u_n (pattern (3)) plus the number of pairs of consecutive bits $(u_{i-1}, u_i) = (1,0)$ in u (pattern (4)). Formally:

$$|\nabla(u)| = u_n + \sum_{i=2}^{n} \max\{u_{i-1} - u_i, 0\}. \quad (7)$$

Next proposition provides us with both the lower and upper shadow of each node u of the intrinsic order graph I_n, using decimal representation.

Proposition 4.1: Let $n \geq 1$, and let $u \in \{0,1\}^n$ with Hamming weight m. Write $u_{(10}$ as sum of powers of 2, in increasing order of the exponents, i.e.,

$$u_{(10} = \sum_{i=1}^{n} 2^{n-i} u_i = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_m} \tag{8}$$

$$(0 \leq q_1 < q_2 < \cdots < q_m \leq n-1).$$

(i) The lower shadow $\Delta(u)$ of u is characterized as follows:
(i)-(a) If $u_{(10}$ is even (i.e., if $u_n = 0$) then

$$u_{(10} + 1 \in \Delta(u), \text{ i.e., } u_{(10} \rhd u_{(10} + 1.$$

(i)-(b) For any power 2^q $(0 \leq q \leq n-2)$ in (8) s.t. 2^{q+1} does not appear in (8) then

$$u_{(10} + 2^q \in \Delta(u), \text{ i.e., } u_{(10} \rhd u_{(10} + 2^q.$$

(ii) The upper shadow $\nabla(u)$ of u is characterized as follows:
(ii)-(a) If $u_{(10}$ is odd (i.e., if $u_n = 1$) then

$$u_{(10} - 1 \in \nabla(u), \text{ i.e., } u_{(10} - 1 \rhd u_{(10}.$$

(ii)-(b) For any power 2^q $(1 \leq q \leq n-1)$ in (8) s.t. 2^{q-1} does not appear in (8) then

$$u_{(10} - 2^{q-1} \in \nabla(u), \text{ i.e., } u_{(10} - 2^{q-1} \rhd u_{(10}.$$

Proof: The assertions (i)-(a) and (ii)-(a) immediately follow using pattern (3) in Theorem 2.2, for matrices M_v^u and M_u^v, respectively. The assertions (i)-(b) and (ii)-(b) immediately follow using pattern (4) in Theorem 2.2, for matrices M_v^u and M_u^v, respectively. ■

B. Neighbors and Degrees

The neighbors of a given vertex u in a graph, are all those nodes adjacent to u (i.e., connected by one edge to u). In particular, for (the cover graph of) a Hasse diagram, the neighbors of vertex u either cover u or are covered by u. In other words, denoting by $N(u)$ the set of neighbors of a vertex $u \in \{0,1\}^n$ in the graph I_n, we have

$$N(u) = \Delta(u) \cup \nabla(u) \tag{9}$$

Next proposition provides the total number of neighbors of each node u of the intrinsic order graph I_n, the so-called degree of u, denoted, as usual, by $\delta(u)$.

Proposition 4.2: Let $n \geq 1$ and $u \in \{0,1\}^n$. The degree $\delta(u)$ of u (i.e., the number of neighbors of u) is

$$\delta(u) = 1 + \sum_{i=2}^{n} |u_i - u_{i-1}|. \tag{10}$$

Proof: Using (6), (7) and (9), we immediately obtain

$$\begin{aligned}
\delta(u) &= |N(u)| = |\Delta(u)| + |\nabla(u)| \\
&= (1 - u_n) + \sum_{i=2}^{n} \max\{u_i - u_{i-1}\ ,\ 0\} \\
&+ u_n + \sum_{i=2}^{n} \max\{u_{i-1} - u_i\ ,\ 0\} \\
&= 1 + \sum_{i=2}^{n} \max\{u_i - u_{i-1}\ ,\ u_{i-1} - u_i\} \\
&= 1 + \sum_{i=2}^{n} |u_i - u_{i-1}|,
\end{aligned}$$

as was to be shown. ■

Example 4.1: Let $n = 4$ and $u = (1,0,1,0)$. Then

$$u = (1,0,1,0) \equiv u_{(10} = 2^1 + 2^3 = 10.$$

Using Proposition 4.1-(i), we get (note that $u_{(10} = 10$ is even, i.e., $u_4 = 0$)

$$\Delta(10) = \{10+1\} \cup \{10 + 2^1\} = \{11, 12\}$$

and using Proposition 4.1-(ii), we get

$$\nabla(10) = \{10 - 2^0, 10 - 2^2\} = \{6, 9\}.$$

Thus (see the graph I_4, the right-most one in Fig. 2)

$$N(10) = \Delta(10) \cup \nabla(10) = \{6, 9, 11, 12\}$$

and using (10), we confirm that the cardinality of $N(10)$ is

$$\begin{aligned}
\delta(10) = |N(10)| &= 1 + \sum_{i=2}^{4} |u_i - u_{i-1}| \\
&= 1 + |u_2 - u_1| + |u_3 - u_2| + |u_4 - u_3| \\
&= 1 + |0-1| + |1-0| + |0-1| = 4.
\end{aligned}$$

V. Subgraphs of the Intrinsic Order Graph

A. Some Relevant Subgraphs

A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ whose vertex set is a subset of that of G, and whose set of edges (adjacency relations) is the subset of that of G restricted to V' [1], i.e., $V' \subseteq V$ and $E' = E|_{V'}$. In this subsection, some relevant subgraphs of the intrinsic order graph I_n are studied. These subgraphs are obtained by successive bisections of I_n.

A bisection of a graph is a partition of its vertex set into two subsets with half the vertices each [1]. Hence, Theorem 2.2 provides a bisection of the (edgeless) graph I_n into its two isomorphic (edgeless) subgraphs I_{n-1} and $2^{n-1} + I_{n-1}$

Of course, this bisection process of the edgeless graph I_n can be reiterated by successively partitioning each one of the obtained subgraphs into its top and bottom halves. This iterative bisection process finishes when we have partitioned I_n into 2^n singleton subgraphs (with 1 vertex each), i.e. into its 2^n nodes.

This particular bisection of the intrinsic order graph means that the poset I_n has a "fractal structure": the whole graph has the same "shape" that each one of its two halves, and the same happens with each one of them, and so on, i.e., the poset I_n has the self-similarity property. Figures 2 and 3 illustrate this fact.

Let us set a consistent notation for this iterative bisection process. Recursively bisecting the graph I_n (with 2^n binary n-tuples) is equivalent to recursively bisecting the truth-table for n Boolean variables (with 2^n rows). Since, by construction, the first bit u_1 in all the n-tuples of the first and second half of the truth-table is 0 and 1, respectively, we denote the first and second half of I_n by I_n^0 and I_n^1, respectively. Analogously, since, by construction, the second bit u_2 in all the n-tuples of the first and second half of both halves of the truth table is 0 and 1, respectively, we denote the first and second half of I_n^0 by $I_n^{0,0}$ and $I_n^{0,1}$, respectively; and we denote the first and second half of I_n^1 by $I_n^{1,0}$ and $I_n^{1,1}$, respectively, and so on.

In general, for all $n \geq 1$, for all $1 \leq k \leq n$ and for all k fixed binary digits $\bar{u}_1, \ldots, \bar{u}_k \in \{0,1\}$, we denote by $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ the $\bar{u}_k + 1$-th half of the $\bar{u}_{k-1} + 1$-th half $\ldots$ of the $\bar{u}_1 + 1$-th half of the poset I_n. In other words, $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ can be graphically obtained after k successive bisections of I_n $(1 \leq k \leq n)$ simply by changing the "0" and "1" bits of the vector $(\bar{u}_1, \ldots, \bar{u}_k)$, by the words "first half" and "second half", respectively. Hence, this is the subset of binary n-tuples whose first or left-most k components are fixed, namely $u_1 = \bar{u}_1, \ldots, u_k = \bar{u}_k$; while their last or right-most $n-k$ components, $u_{k+1}, \ldots, u_n$, take all possible values (0 or 1). More precisely, $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ is the set of binary n-tuples

$$\left\{ (\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) \,\middle|\, (u_{k+1}, \ldots, u_n) \in \{0,1\}^{n-k} \right\} \tag{11}$$

or, alternatively, using the decimal representation, $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ is the interval

$$\left[(\bar{u}_1, \ldots, \bar{u}_k, 0, \ldots, 0)_{(10}\,, (\bar{u}_1, \ldots, \bar{u}_k, 1, \ldots, 1)_{(10} \right]. \tag{12}$$

The so obtained graphs $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ are relevant subgraphs of the intrinsic order graph I_n with interesting theoretical properties like, for instance, the ones presented in the next subsection.

The cardinality of these subgraphs are

$$\left| I_n^{\bar{u}_1,\ldots,\bar{u}_k} \right| = \left| \{0,1\}^{n-k} \right| = 2^{n-k}. \tag{13}$$

Remark 5.1: In particular, for $k = n$, the subgraph $I_n^{\bar{u}_1,\ldots,\bar{u}_n}$, obtained after n bisections of I_n, is reduced to a single node of this graph, namely

$$I_n^{\bar{u}_1,\ldots,\bar{u}_n} = \{(\bar{u}_1, \ldots, \bar{u}_n)\} \quad \text{(a curious fact!)}. \tag{14}$$

With this notation, we can formalize the iterative bisection process as follows

$$\begin{aligned} I_n &= I_n^0 \cup I_n^1 = I_n^{0,0} \cup I_n^{0,1} \cup I_n^{1,0} \cup I_n^{1,1} \\ &= I_n^{0,0,0} \cup I_n^{0,0,1} \cup I_n^{0,1,0} \cup I_n^{0,1,1} \\ &\cup \;\; I_n^{1,0,0} \cup I_n^{1,0,1} \cup I_n^{1,1,0} \cup I_n^{1,1,1} \\ &= \cdots = \bigcup_{(\bar{u}_1,\ldots,\bar{u}_n) \in \{0,1\}^n} I_n^{\bar{u}_1,\ldots,\bar{u}_n} \\ &= \bigcup_{(\bar{u}_1,\ldots,\bar{u}_n) \in \{0,1\}^n} \{(\bar{u}_1, \ldots, \bar{u}_n)\}. \end{aligned} \tag{15}$$

Example 5.1: For the graph of I_3 (the third one from the left in Fig. 2), using (15), we have

$$\begin{aligned} I_3 &= [0,7] = [0,3] \cup [4,7] = [0,1] \cup [2,3] \cup [4,5] \cup [6,7] \\ &= \{0\} \cup \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\} \cup \{7\}. \end{aligned}$$

Example 5.2: For $n = 5$, $k = 3$ and for the binary 3-tuple $(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (0,1,1)$, we get the subgraph

$$\begin{aligned} I_5^{0,1,1} &= \left\{ (0,1,1,u_4,u_5) \,\middle|\, (u_4,u_5) \in \{0,1\}^2 \right\} \\ &= \left[2^2 + 2^3, 2^0 + 2^1 + 2^2 + 2^3 \right] = [12,15] \\ &= \{12, 13, 14, 15\} \end{aligned}$$

and looking at the fifth diagram from the left in Fig. 3, we confirm that $[12,15]$ is exactly the second half $(\bar{u}_3 = 1)$ of the second half $(\bar{u}_2 = 1)$ of the first half $(\bar{u}_1 = 0)$ of the poset I_5. In accordance with (13), $I_5^{0,1,1}$ has $2^{5-3} = 4$ elements.

Example 5.3: For $n = 6$, for $k = 6$ and for the binary 6-tuple $(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5, \bar{u}_6) = (1,0,1,0,1,0)$, using (14) –here $k = n$–, we get the singleton subgraph

$$I_6^{1,0,1,0,1,0} = \{(1,0,1,0,1,0)\} = \left\{ 2^1 + 2^3 + 2^5 \right\} = \{42\}$$

and looking at the right-most diagram in Fig. 3, we confirm that $\{42\}$ is exactly the first half $(\bar{u}_6 = 0)$ of the second half $(\bar{u}_5 = 1)$ of the first half $(\bar{u}_4 = 0)$ of the second half $(\bar{u}_3 = 1)$ of the first half $(\bar{u}_2 = 0)$ of the second half $(\bar{u}_1 = 1)$ of the poset I_6. In accordance with (13), $I_6^{1,0,1,0,1,0}$ has $2^{6-6} = 1$ element.

B. Isomorphisms of Subgraphs

Let $n \geq 1$ and $1 \leq k \leq n$. Let $\bar{u}_1, \ldots, \bar{u}_k \in \{0,1\}$ be k fixed binary digits. Let $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ be the subgraph of I_n defined by (11) or by (12).

Let us recall that two graphs $G(V,E)$ and $G^*(V^*,E^*)$ are said to be isomorphic if there exists an isomorphism of one of them to the other, i.e., an edge-preserving bijection [1]. That is, a graph isomorphism is a one-to-one mapping between the vertex sets $\Phi : V \to V^*$, which preserves adjacency, i.e., u, v are adjacent in G if and only if $\Phi(u), \Phi(v)$ are adjacent in G^*.

The self-similarity property or fractal structure that one can observe in Figs. 2 & 3, is an immediate consequence of the following two propositions.

Proposition 5.1: Let $n \geq 1$ and $1 \leq k \leq n$. The 2^k equal-sized subgraphs $I_n^{\bar{u}_1,\ldots,\bar{u}_k}$ (each with 2^{n-k} nodes), obtained after k successive bisections of the intrinsic order graph I_n, are pair-wise isomorphic, and indeed all of them are isomorphic to the intrinsic order graph I_{n-k}.

Proof: Consider the following mapping

$$\begin{array}{ccc} I_n^{\bar{u}_1,\ldots,\bar{u}_k} & \xrightarrow{\Phi} & I_{n-k} \\ (\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) & \longmapsto & (u_{k+1}, \ldots, u_n). \end{array}$$

Obviously Φ is a one-to-one mapping. Moreover, using Theorem 2.2, we have

$$(\bar{u}_1, \ldots, \bar{u}_k, u_{k+1}, \ldots, u_n) \rhd (\bar{u}_1, \ldots, \bar{u}_k, v_{k+1}, \ldots, v_n)$$

if and only if matrix

$$\begin{pmatrix} \bar{u}_1 & \ldots & \bar{u}_k & u_{k+1} & \ldots & u_n \\ \bar{u}_1 & \ldots & \bar{u}_k & v_{k+1} & \ldots & v_n \end{pmatrix}$$

has either the pattern (3) or the pattern (4) if and only if matrix

$$\begin{pmatrix} u_{k+1} & \ldots & u_n \\ v_{k+1} & \ldots & v_n \end{pmatrix}$$

has either the pattern (3) or the pattern (4) if and only if

$$(u_{k+1}, \ldots, u_n) \rhd (v_{k+1}, \ldots, v_n),$$

so that Φ is an isomorphism of graphs, since it preserves the edges (covering relations). ■

For instance, let $n = 5$ and $k = 3$. After $k = 3$ successive bisections of the intrinsic order graph I_5, the $2^k = 8$ subgraphs are the 8 isomorphic "columns" or, more formally, congruence chains $(\bmod 4)$ (each containing $2^{n-k} = 4$ nodes) depicted in Fig. 4. Moreover, any of these "column"-subgraphs of I_5 (5-tuples) is isomorphic to I_2 (2-tuples), the second graph from the left in Fig. 2.

The fractal structure of the intrinsic order graph is not only a consequence of Proposition 5.1, but it is also a consequence of the following result.

Proposition 5.2: Let $n \geq 1$ and $1 \leq k \leq n$. Bisect the edgeless graph I_n into its 2^k subgraphs $I_n^{\bar{u}_1,\dots,\bar{u}_k}$ (i.e. make k successive bisections of I_n). Replace each subgraph $I_n^{\bar{u}_1,\dots,\bar{u}_k}$ by an unique node labeled by its corresponding vector of upper indices $(\bar{u}_1,\dots,\bar{u}_k)$ and weighted by the occurrence probability $\Pr\{(\bar{u}_1,\dots,\bar{u}_k)\}$ of its label. Next, sort these 2^k new nodes in decreasing order of their weights. Then the new "condensed" graph obtained from the intrinsic order graph I_n –with 2^n vertices– by this "bisecting-replacing-sorting" process, is precisely the intrinsic order graph I_k –with 2^k vertices. Moreover, this ordering between the 2^k new nodes coincides with the ordering between the 2^k sums of the occurrence probabilities of all the nodes lying on each one of the respective replaced subgraphs.

Proof: Sorting the 2^k vertices of the new graph in decreasing order of their assigned weights $\Pr\{(\bar{u}_1,\dots,\bar{u}_k)\}$ is equivalent to ordering the 2^k binary k-tuples $(\bar{u}_1,\dots,\bar{u}_k) \in \{0,1\}^k$ in decreasing order of their occurrence probabilities. Thus, the new condensed graph is, by definition, the intrinsic order graph I_k. Finally, using the obvious fact that

$$\sum_{(u_{k+1},\dots,u_n)\in\{0,1\}^n} \Pr\{(u_{k+1},\dots,u_n)\} = 1$$

we get

$$\Pr\left\{I_n^{\bar{u}_1,\dots,\bar{u}_k}\right\} = \sum_{u\in I_n^{\bar{u}_1,\dots,\bar{u}_k}} \Pr\{u\} = \Pr\{(\bar{u}_1,\dots,\bar{u}_k)\},$$

and this proves the last statement of the theorem. as was to be shown. ■

The statement of Proposition 5.2 can be summed up by the following sentence: k successive bisections of the digraph I_n lead to the digraph I_k. In Fig. 5 this proposition is illustrated for $n = 5$ and $k = 1, 2, 3$. Note that wile the nodes of I_5 are binary 5-tuples, the vertices of the corresponding graphs I_1, I_2 and I_3 are binary 1-tuples, 2-tuples and 3-tuples, respectively.

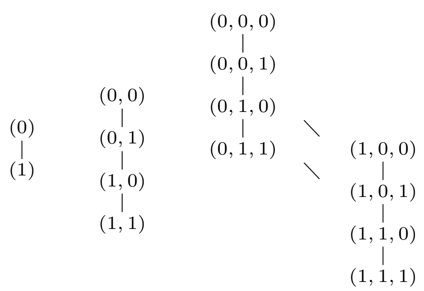

Fig. 5. k successive bisections of the digraph I_n lead to the digraph I_k ($n = 5$, $k = 1, 2, 3$).

Corollary 5.1: Let $n \geq 1$ and $1 \leq k \leq n$. Then the subgraphs $I_n^{0,\overset{k}{\dots},0}$ and $I_n^{1,\overset{k}{\dots},1}$ are the ones with the largest and smallest occurrence probabilities (i.e., sum of the occurrence probabilities of all nodes lying on each of them), respectively, among all the 2^k subgraphs $I_n^{\bar{u}_1,\dots,\bar{u}_k}$ obtained after k successive bisections of I_n.

Proof: Using Theorem 5.2, we see that proving the current theorem is equivalent to proving that, for all $k \geq 1$, the binary k-tuples

$$\left(0,\overset{k}{\dots},0\right) = 0 \quad \text{and} \quad \left(1,\overset{k}{\dots},1\right) = 2^k - 1$$

are the maximum and minimum elements, respectively, in the poset I_k. This fact, illustrated by Figs. 2 & 3, has been demonstrated in Example 2.5. ■

VI. Conclusion

In this paper, we have considered complex systems depending on an arbitrarily large number n of random Boolean variables, i.e., the so-called complex stochastic Boolean systems (CSBSs). We have defined and characterized by a simple matrix description the intrinsic order between the binary n-tuples associated to a CSBS. Then we have presented the usual graphical representation for CSBSs: a Hasse diagram on 2^n nodes called the intrinsic order graph, and denoted by I_n. New properties of the intrinsic order graph have been stated and proved. These properties deal with different features of the intrinsic order graph like, e.g., its edges; the natural decomposition of the graph I_n into its 2^{n-2} "columns of size 4" or congruence chains $(\bmod 4)$; the shadows, neighbors and degrees of its vertices; and the study of some relevant isomorphic subgraphs of I_n obtained by bisection. From a theoretical point of view, this paper suggests the search of new graph-theoretic and order-theoretic properties of the intrinsic order graph I_n. For practical applications, some of these properties can be applied to develop new algorithms that identify binary strings with large occurrence probabilities. Such algorithms can be used in Reliability Theory and Risk Analysis to estimate the failure probability of a technical system modeled by a CSBS.

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