Analysis of several objective functions for optimization of hexahedral meshes

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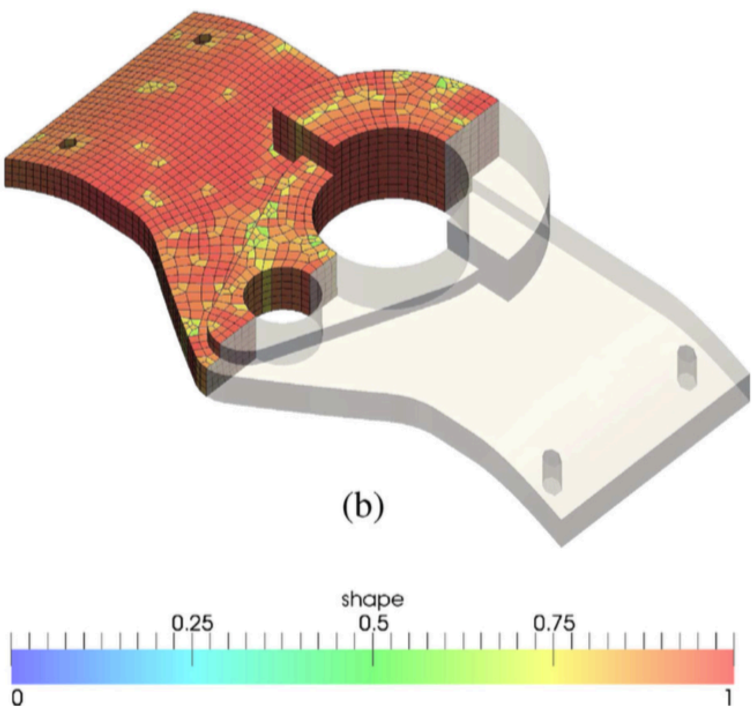
Optimization of hexahedral meshes

The quality of the mesh has high repercussion on the numerical behaviour of FEM

All elements should be valid



(b)

E. Ruiz-Gironés, X. Roca, J. Sarrate, R. Montenegro, J.M. Escobar, **Simultaneous untangling and smoothing of quadrilateral and hexahedral meshes using an object-oriented framework**, Advances in Engineering Software 80 (2015) 12-24

What is a valid element?

2D element validity

Quadrilateral element

For FEM, one to one transformation is required

Reference element

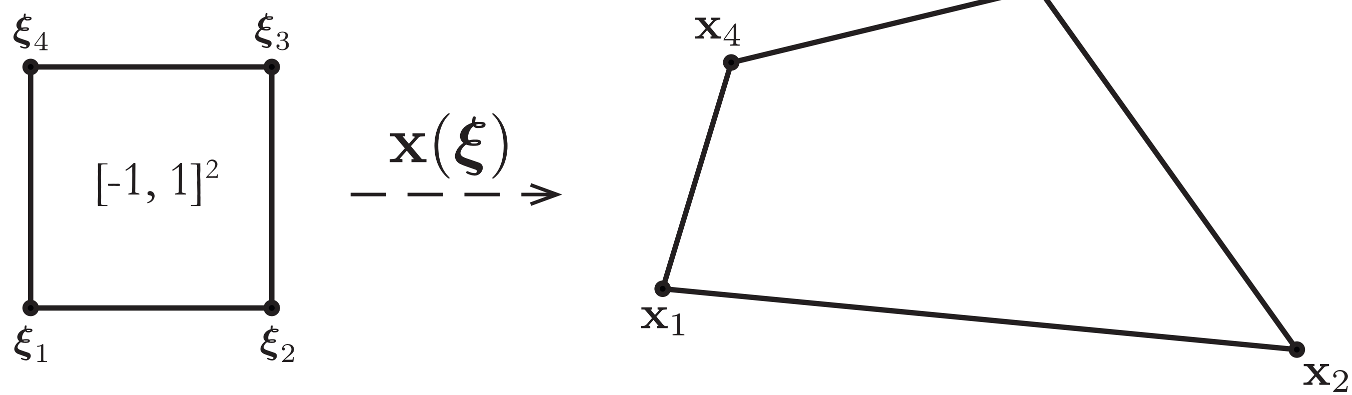
Physical element

$\xi_1 = (-1, -1)$

$\xi_2 = (1, -1)$

$\xi_3 = (1, 1)$

$\xi_4 = (-1, 1)$



Bilinear transformation

$$\mathbf{x}(\boldsymbol{\xi}) = (x(\xi,\eta), y(\xi,\eta)) = \sum_{i=1}^{4} \mathbf{x}_i N_i(\boldsymbol{\xi})$$

$$N_i(\boldsymbol{\xi}) = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

valid if

Positive Jacobian in the whole element

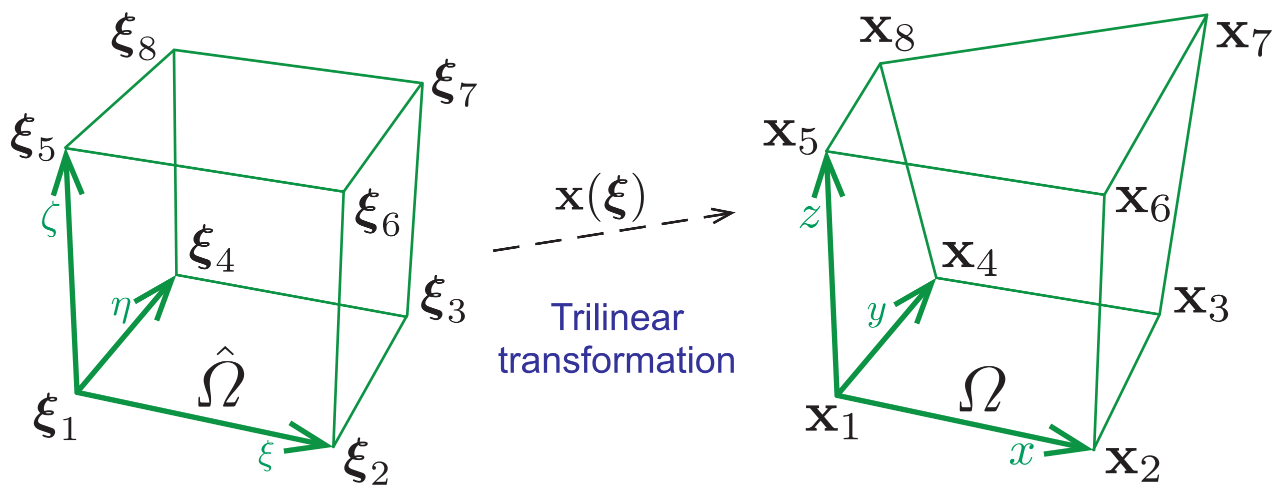
$$J(\boldsymbol{\xi}) = \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right)$$

$$J(\boldsymbol{\xi}) > 0, \quad \boldsymbol{\xi} \in [-1, 1]^2$$

3D element validity

Hexahedral element

$$
\begin{aligned}
\boldsymbol{\xi}_1 &= (-1,-1,-1) \\
\boldsymbol{\xi}_2 &= (1,-1,-1) \\
\boldsymbol{\xi}_3 &= (1,1,-1) \\
\boldsymbol{\xi}_4 &= (-1,1,-1) \\
\boldsymbol{\xi}_5 &= (-1,-1,1) \\
\boldsymbol{\xi}_6 &= (1,-1,1) \\
\boldsymbol{\xi}_7 &= (1,1,1) \\
\boldsymbol{\xi}_8 &= (-1,1,1)
\end{aligned}
$$



valid element →

$$J(\boldsymbol{\xi}) = \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right)$$

$$J(\boldsymbol{\xi}) > 0,\ \boldsymbol{\xi} \in [-1,1]^3$$

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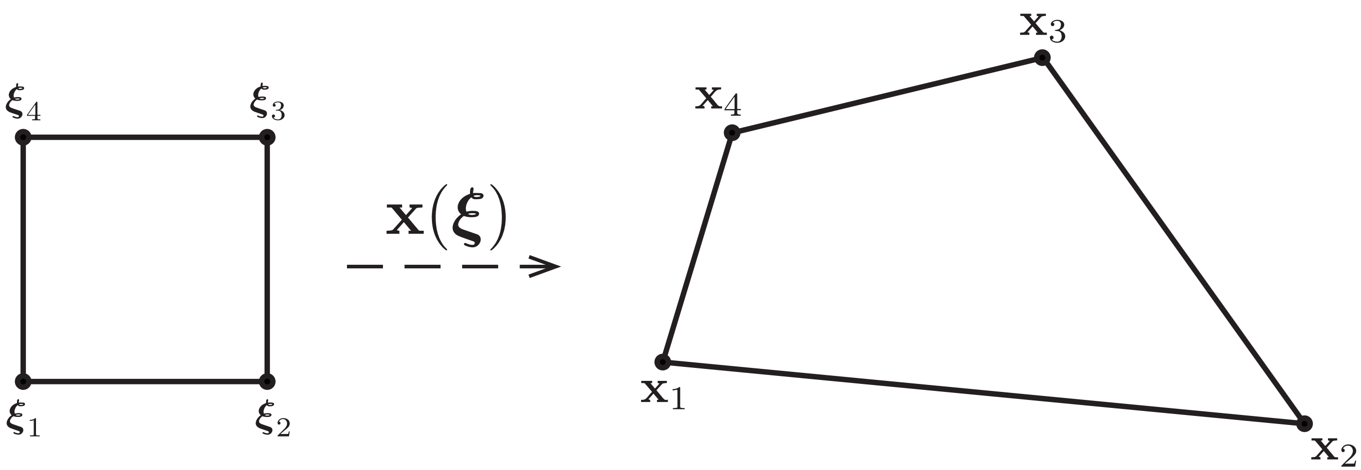
What conditions do ensure positive Jacobian?

Jacobian positivity of quadrilaterals

Subdivide the element in four triangles

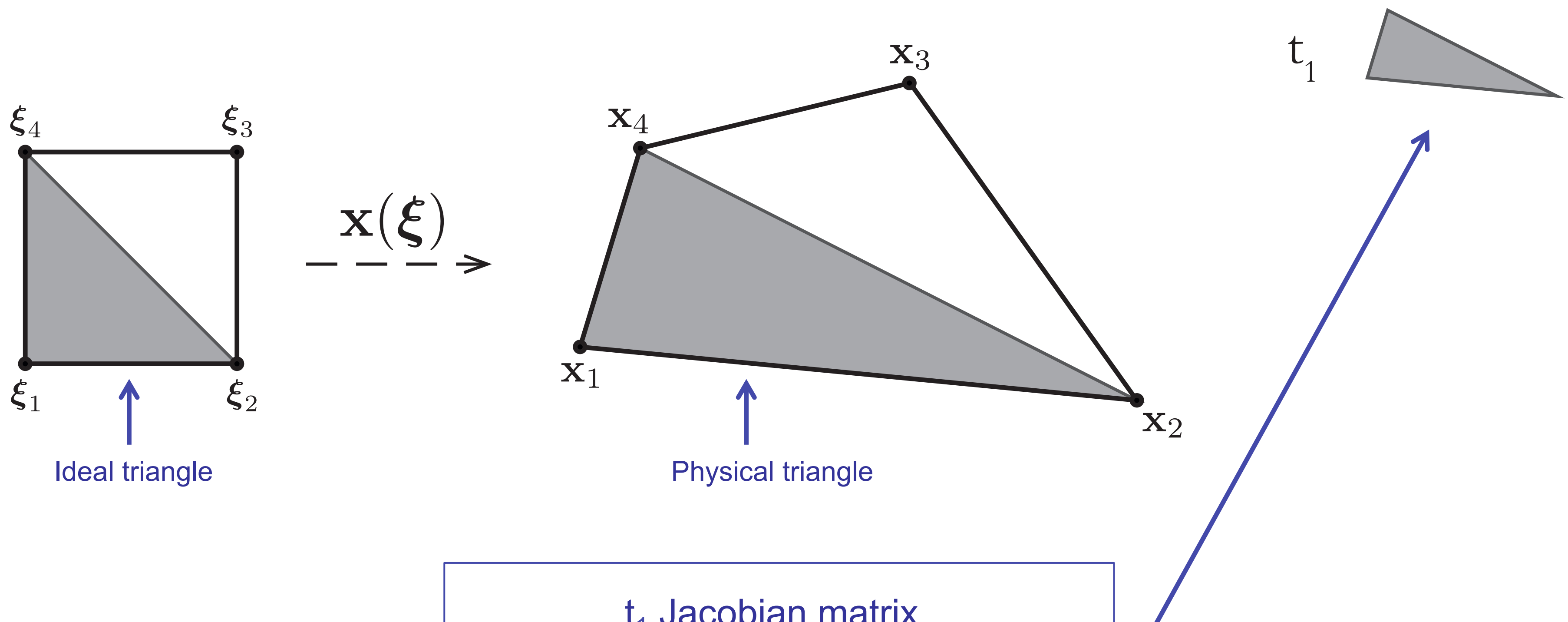


$$J(\boldsymbol{\xi}) > 0, \ \ \boldsymbol{\xi} \in [-1, 1]^2$$

$$J(\boldsymbol{\xi}) = \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right)$$

Jacobian positivity of quadrilaterals

Subdivide the element in four triangles

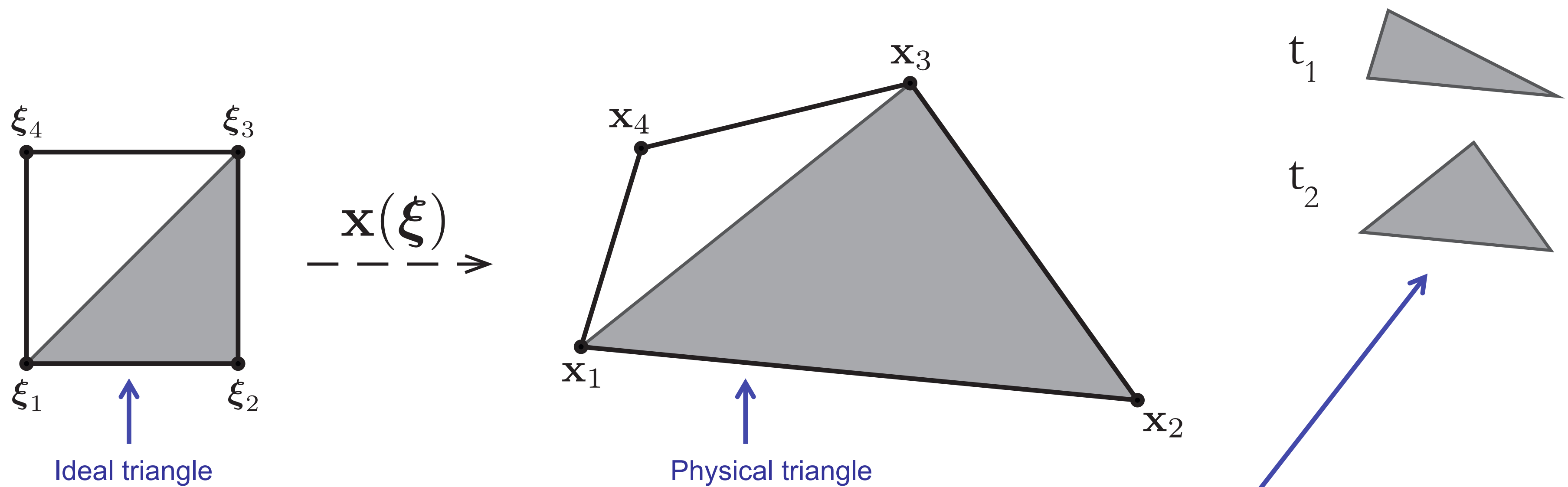


t_1 Jacobian matrix

$$A_1 = (\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_1)$$

Jacobian positivity of quadrilaterals

Subdivide the element in four triangles

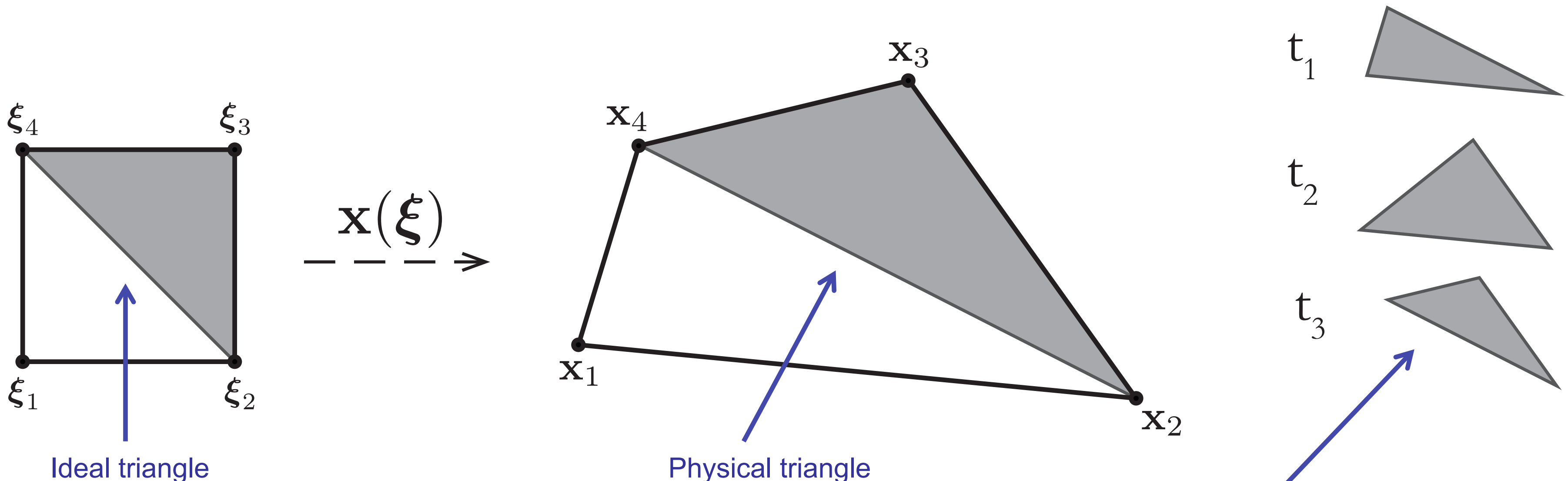


t_2 Jacobian matrix

$$A_2 = (\mathbf{x}_3 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2)$$

Jacobian positivity of quadrilaterals

Subdivide the element in four triangles

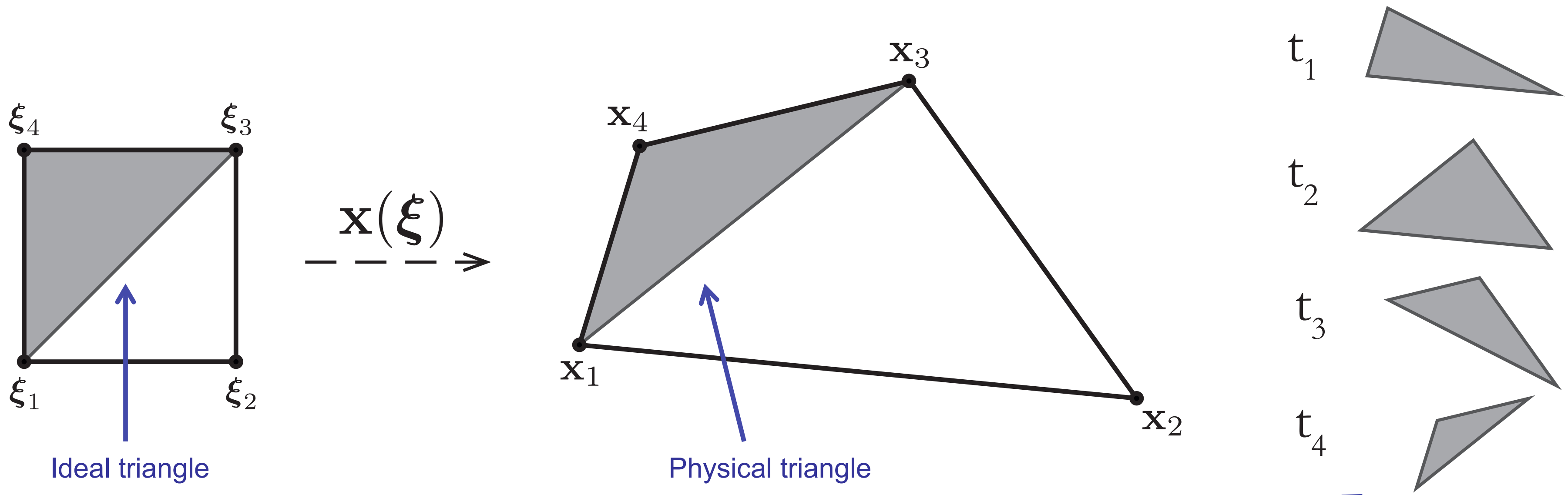


t_3 Jacobian matrix

$$A_3 = (\mathbf{x}_4 - \mathbf{x}_3, \mathbf{x}_2 - \mathbf{x}_3)$$

Jacobian positivity of quadrilaterals

Subdivide the element in four triangles

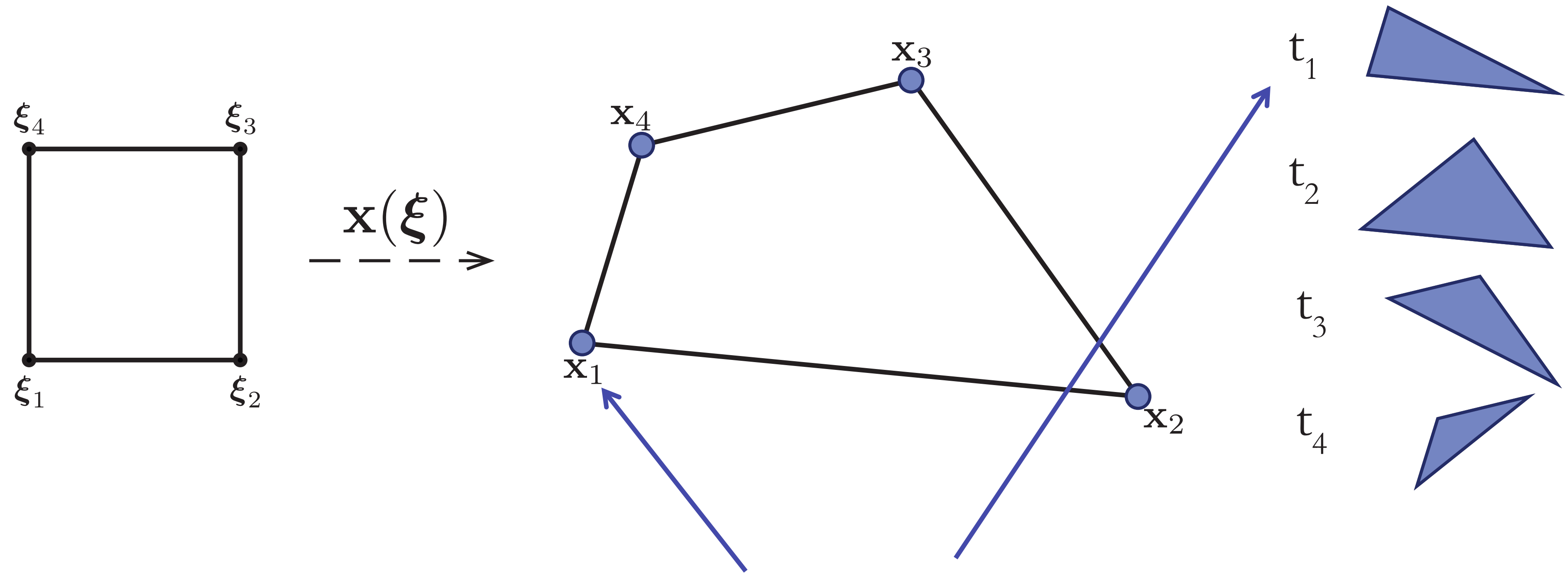


t_4 Jacobian matrix

$$A_4 = (\mathbf{x}_1 - \mathbf{x}_4, \mathbf{x}_3 - \mathbf{x}_4)$$

Jacobian positivity of quadrilaterals

Positive area of the triangles entail positive Jacobian of the bilinear transformation at vertices

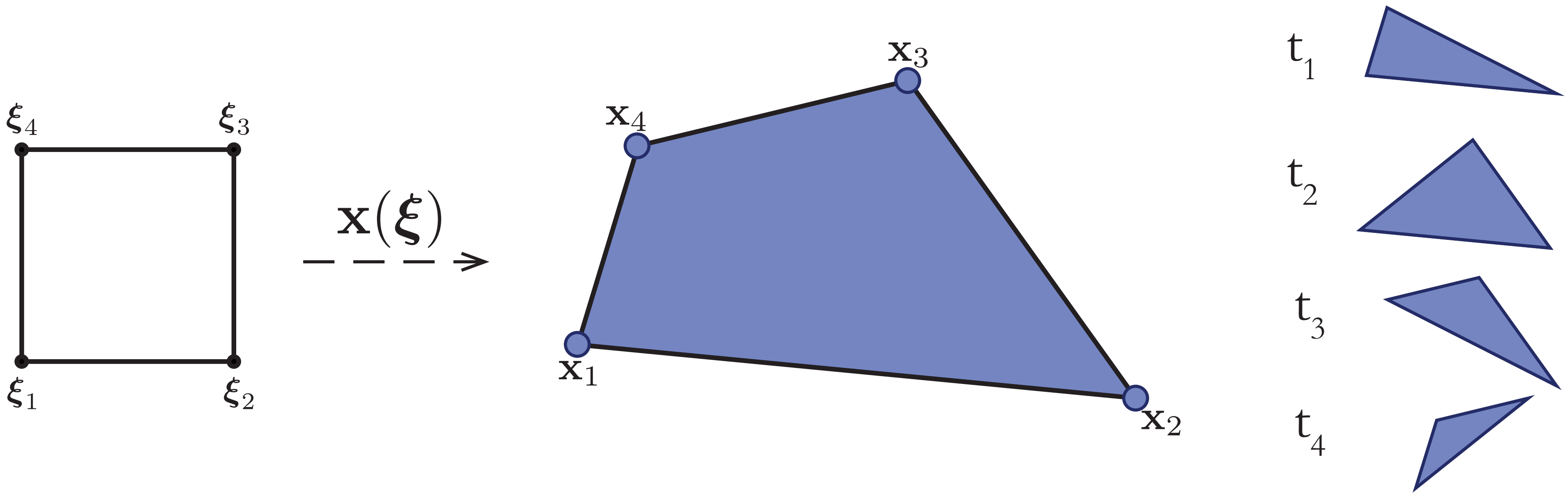


$$\det(A_i) > 0 \Longleftrightarrow J(\boldsymbol{\xi}_i) > 0$$
$$i = 1, \ldots, 4$$

Necessary condition

Jacobian positivity of quadrilaterals

Positive Jacobian at vertices is **necessary and sufficient** condition for positive Jacobian in the whole element



Sufficient condition

$$\det(A_i) > 0 \Longleftrightarrow \boxed{J(\boldsymbol{\xi}_i) > 0 \Longrightarrow \underset{\boldsymbol{\xi} \in [-1,1]^3}{J(\boldsymbol{\xi}) > 0}}$$

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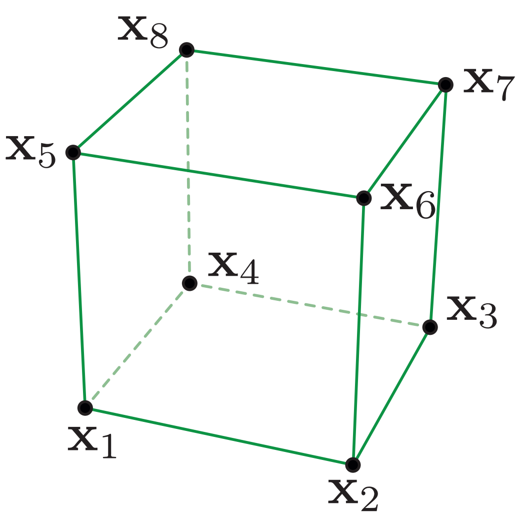
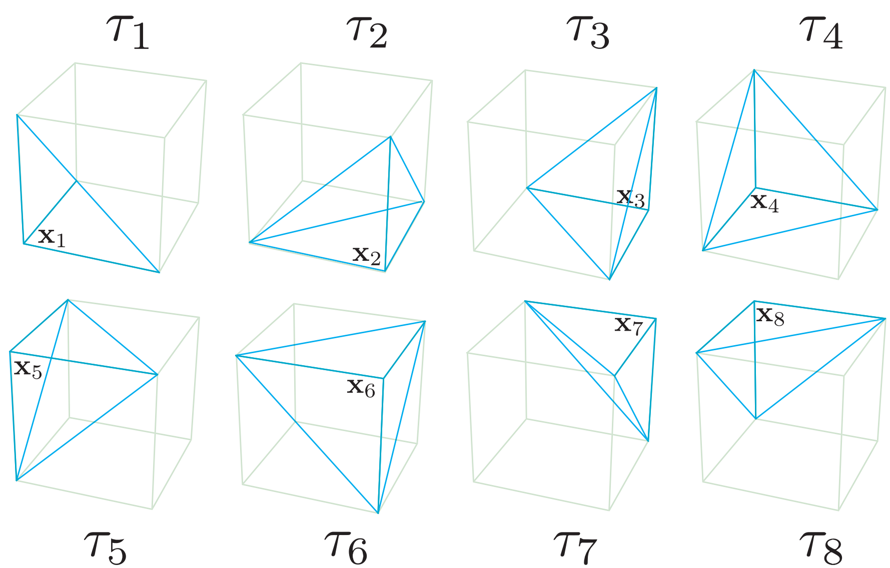
And for a hexahedron?

Jacobian positivity of hexahedra



Subdivide the element in eight tetrahedra

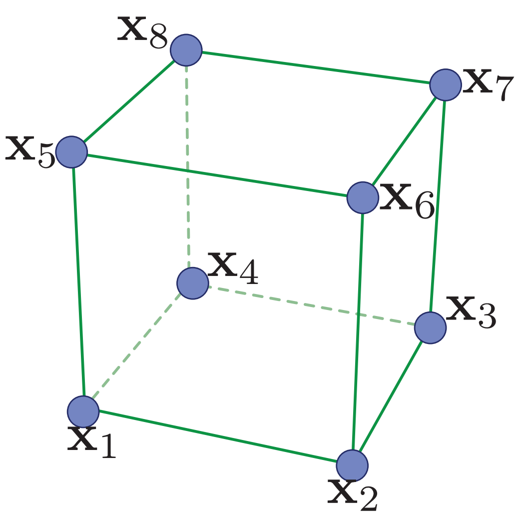
$$\tau_i = (\mathbf{x}_1^i, \mathbf{x}_2^i, \mathbf{x}_3^i, \mathbf{x}_4^i)$$
$$i = 1, ..., 8$$



Tetrahedron Jacobian matrix

$$A_i = (\mathbf{x}_2^i - \mathbf{x}_1^i, \mathbf{x}_3^i - \mathbf{x}_1^i, \mathbf{x}_4^i - \mathbf{x}_1^i)$$

Jacobian positivity of hexahedra

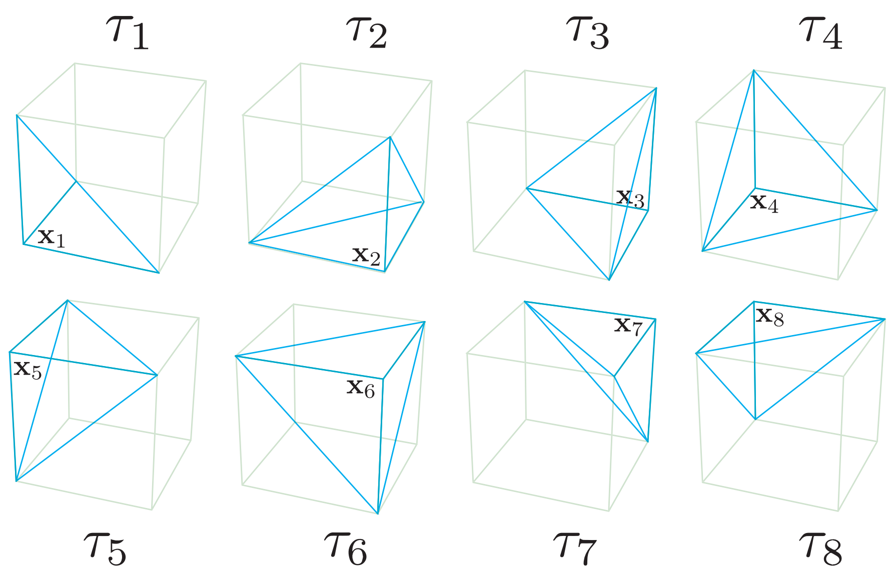


Positive volume of each tetrahedra entail positive Jacobian of the trilinear transformation at vertices

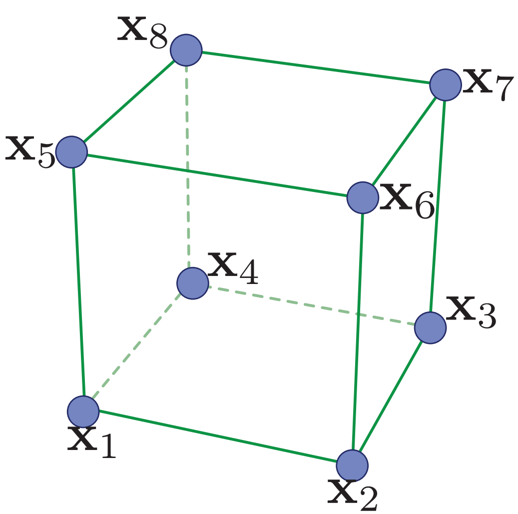
$$\boxed{\det(A_i) > 0 \Longleftrightarrow J(\boldsymbol{\xi}_i) > 0} \Longrightarrow J(\boldsymbol{\xi}) > 0$$

$$\boldsymbol{\xi} \in [-1, 1]^3$$

Necessary condition



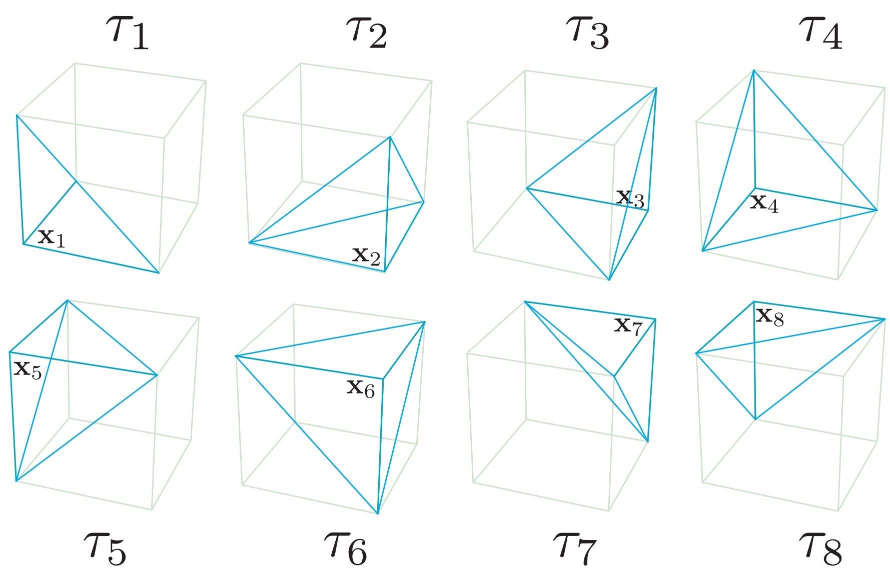
Jacobian positivity of hexahedra



Does it ensure positive Jacobian in the whole element?

$$\det(A_i) > 0 \iff \boxed{J(\boldsymbol{\xi}_i) > 0 \Longrightarrow \underset{\boldsymbol{\xi} \in [-1,1]^3}{J(\boldsymbol{\xi}) > 0}}$$

Sufficient condition?

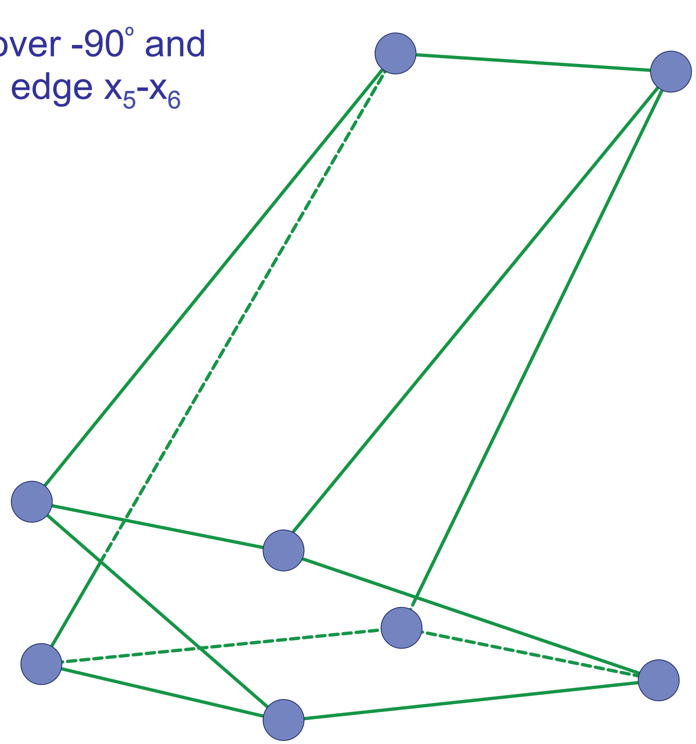


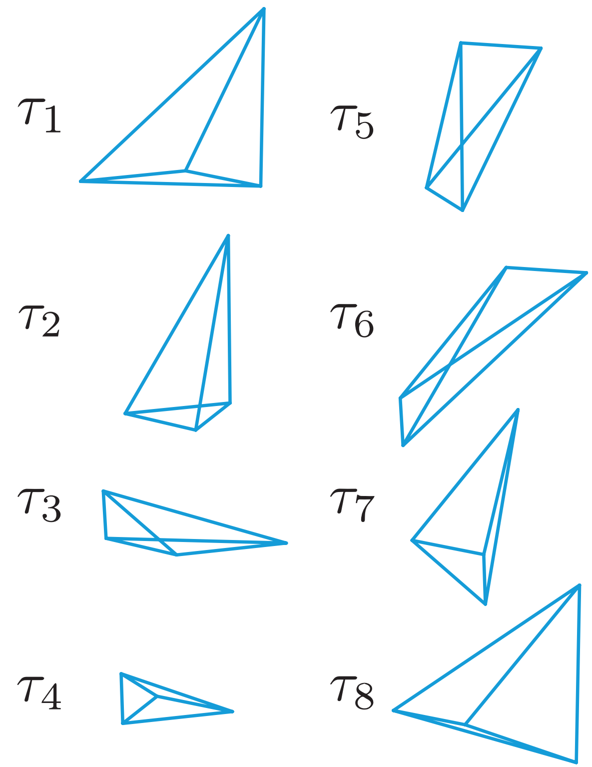
See an example

Jacobian positivity of hexahedra

rotate top cover -90° and elevate +5 edge x_5-x_6

$$
\begin{aligned}
\mathbf{x}_1 &= (-1,-1,-1)\\
\mathbf{x}_2 &= (1,-1,-1)\\
\mathbf{x}_3 &= (1,\ 1,-1)\\
\mathbf{x}_4 &= (-1,\ 1,-1)\\
\mathbf{x}_5 &= (-1,\ 1,\ 6)\\
\mathbf{x}_6 &= (-1,-1,\ 6)\\
\mathbf{x}_7 &= (1,-1,\ 1)\\
\mathbf{x}_8 &= (1,\ 1,\ 1)
\end{aligned}
$$





Positive Jacobian at vertices

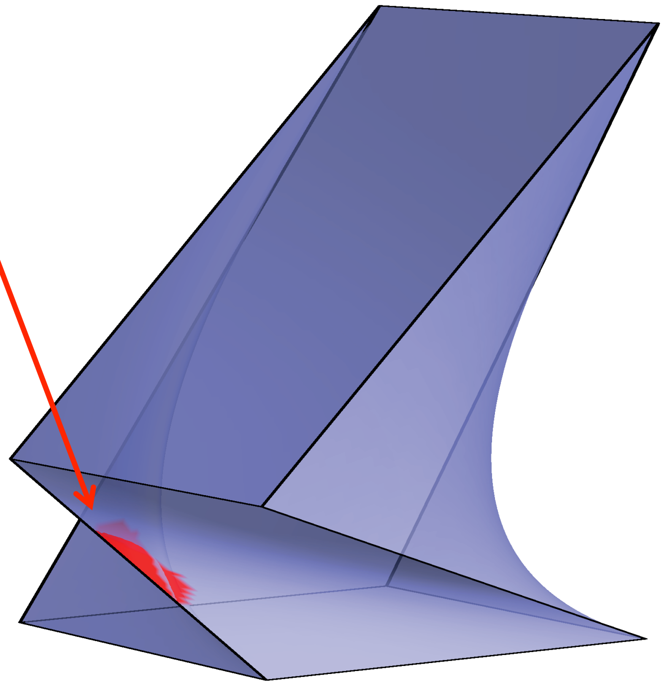
8 tets are valid

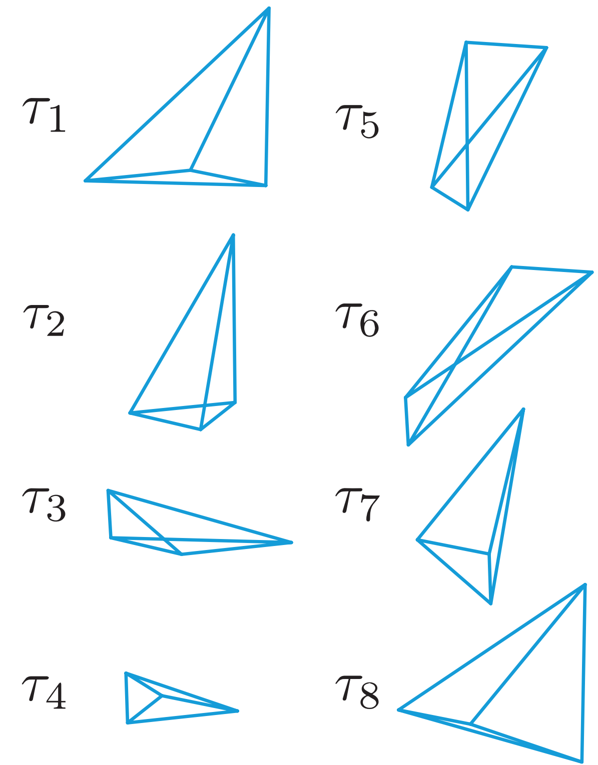
$$\det(A_i) > 0 \iff J(\boldsymbol{\xi}_i) > 0$$
$$i = 1, \ldots, 8$$

Jacobian positivity of hexahedra

Negative Jacobian !

$$
\begin{aligned}
\mathbf{x}_1 &= (-1,-1,-1) \\
\mathbf{x}_2 &= (1,-1,-1) \\
\mathbf{x}_3 &= (1,\ 1,-1) \\
\mathbf{x}_4 &= (-1,\ 1,-1) \\
\mathbf{x}_5 &= (-1,\ 1,\ 6) \\
\mathbf{x}_6 &= (-1,-1,\ 6) \\
\mathbf{x}_7 &= (1,-1,\ 1) \\
\mathbf{x}_8 &= (1,\ 1,\ 1)
\end{aligned}
$$





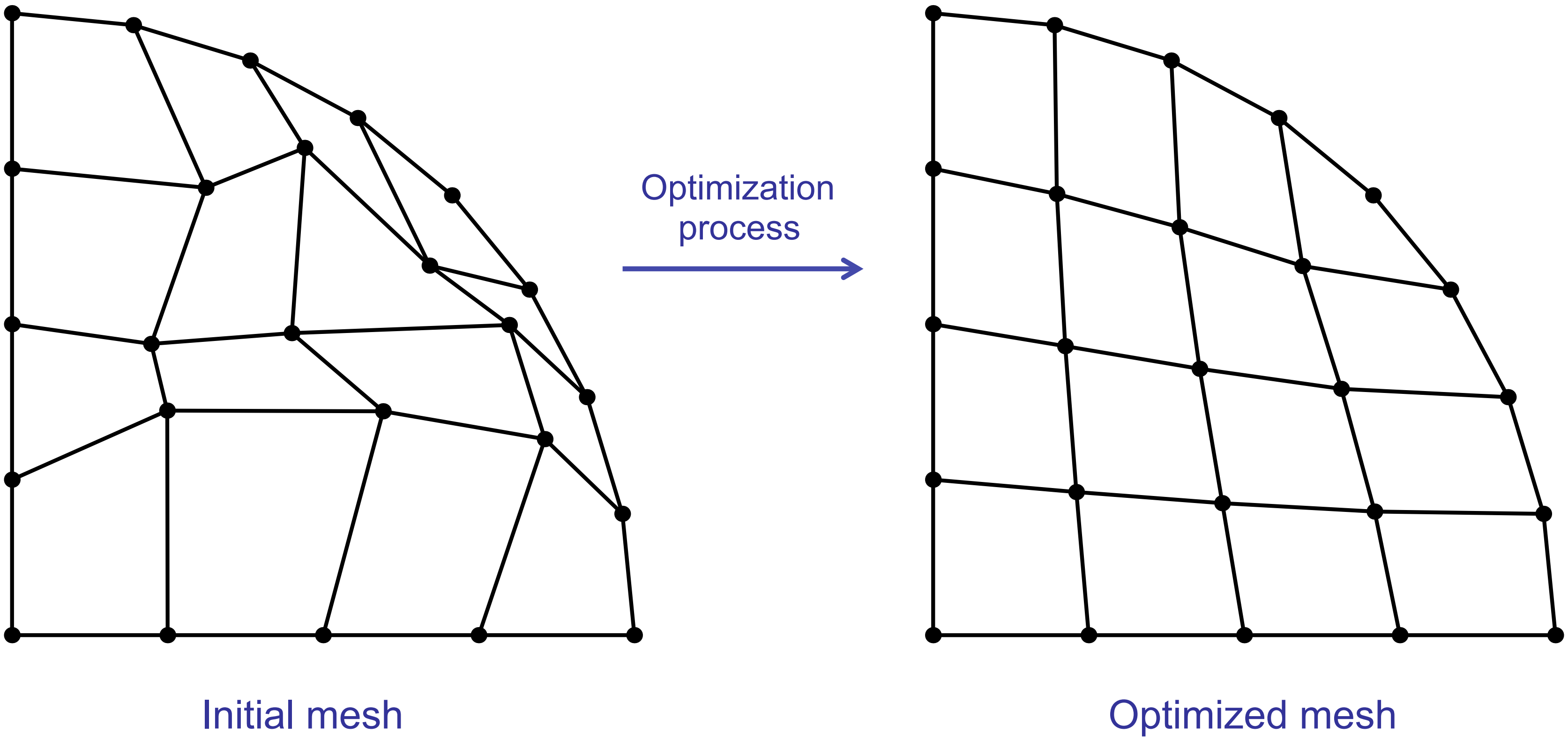
$$\det(A_i) > 0 \Longleftrightarrow \boxed{J(\boldsymbol{\xi}_i) > 0 \not\Longrightarrow J(\boldsymbol{\xi}) > 0}$$

Not sufficient condition

Local optimization

Local optimization

Goal: to obtain a valid mesh and improve the quality of the elements

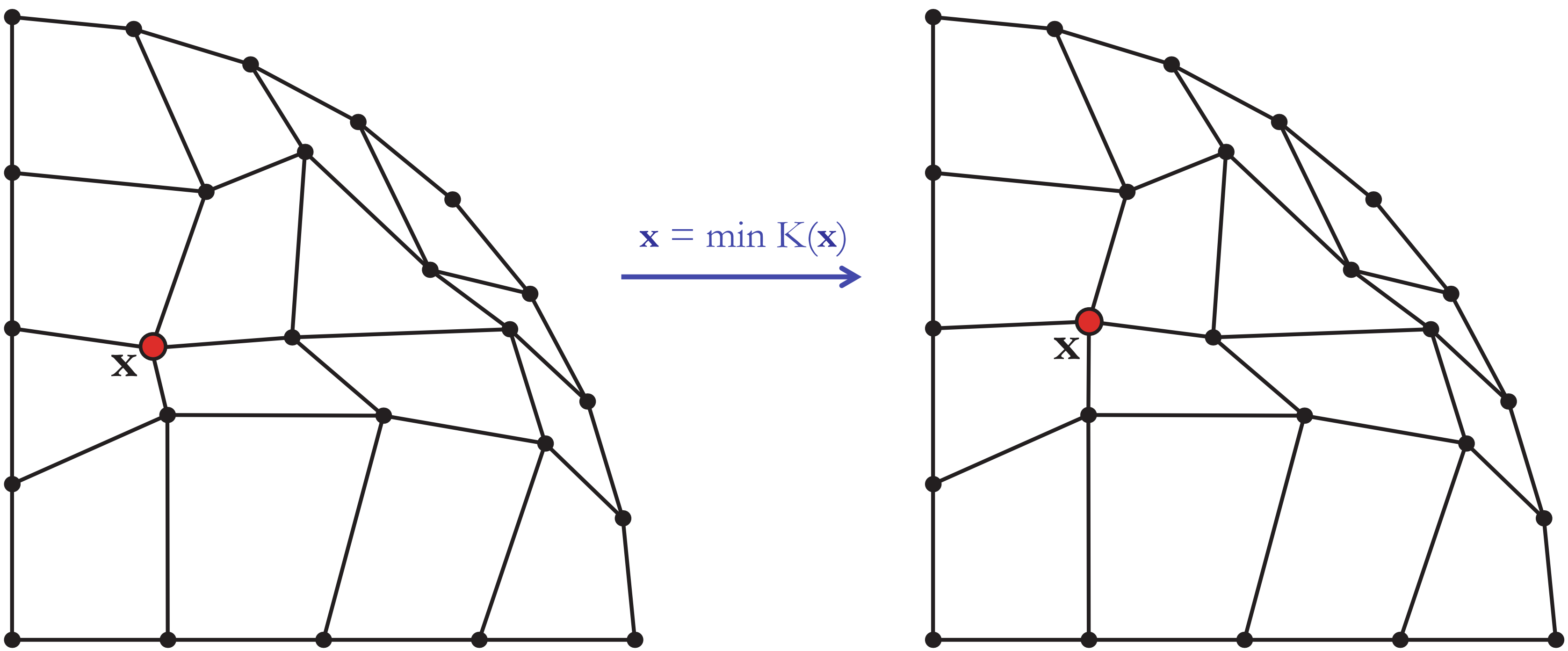


Local optimization

Iterative process

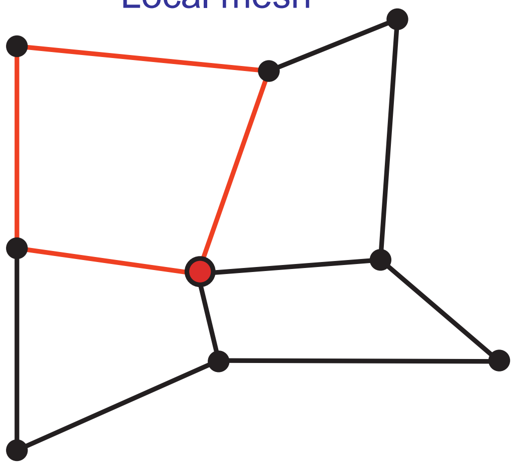
Each node is moved to a new position

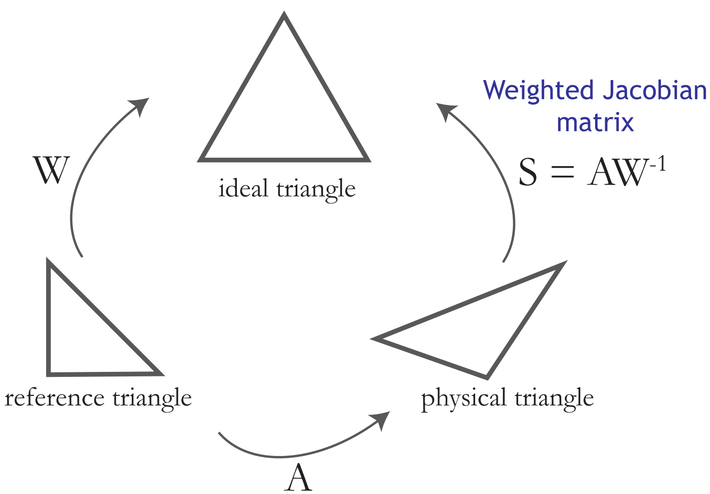
Minimize an objective function

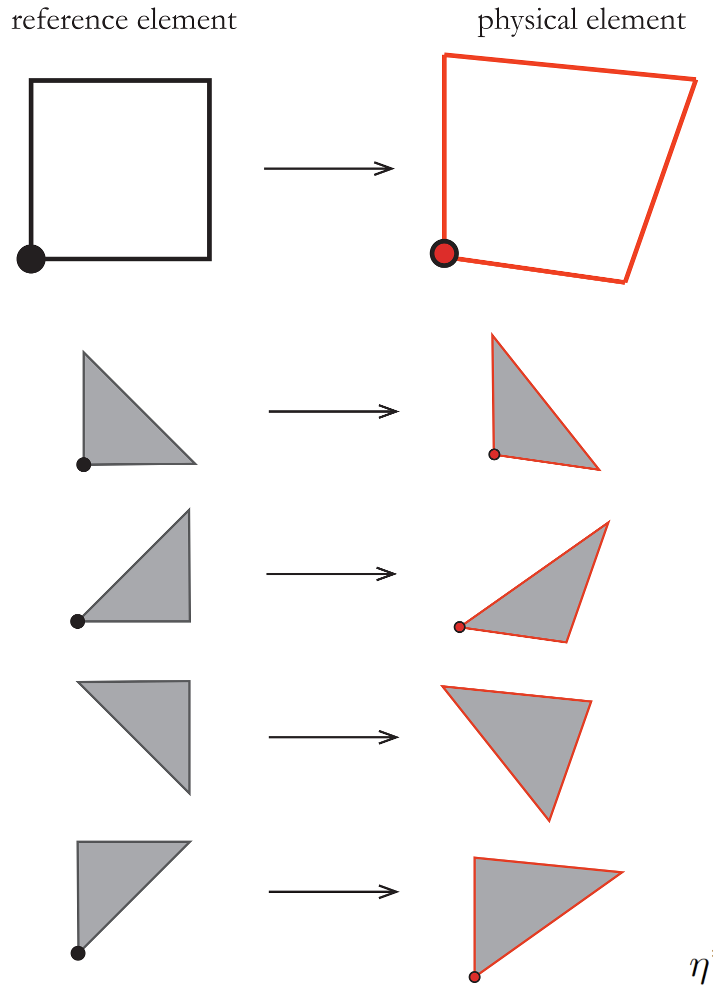


Objective function for quadrilaterals

Local mesh







$$\sigma = \det(S)$$

$\|S\|$ is the Frobenius norm of S

Shape quality measure of a triangle (mean ratio)

$$q(S) = \frac{2\sigma}{\|S\|^2} = \frac{1}{\eta}$$

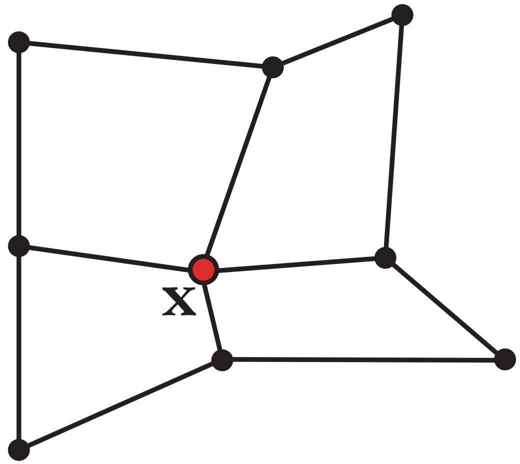
Regularized shape distortion measure

$$\eta^*(S) = \frac{\|S\|^2}{2h(\sigma)}$$

Distortion of a quadrilateral

$$\eta^*(\mathbf{x}) = \left[\frac{1}{4}\sum_{i=1}^{4}(\eta^*(S_i))^p\right]^{1/p}$$

Objective function for quadrilaterals



$$\sigma = \det(S)$$

$\|S\|$ is the Frobenius norm of S

Shape quality measure of a triangle (mean ratio)

$$q(S) = \frac{2\sigma}{\|S\|^2} = \frac{1}{\eta}$$

Objective Function

$$\boxed{K^*(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N}(\eta_i^*(\mathbf{x}))^p}$$

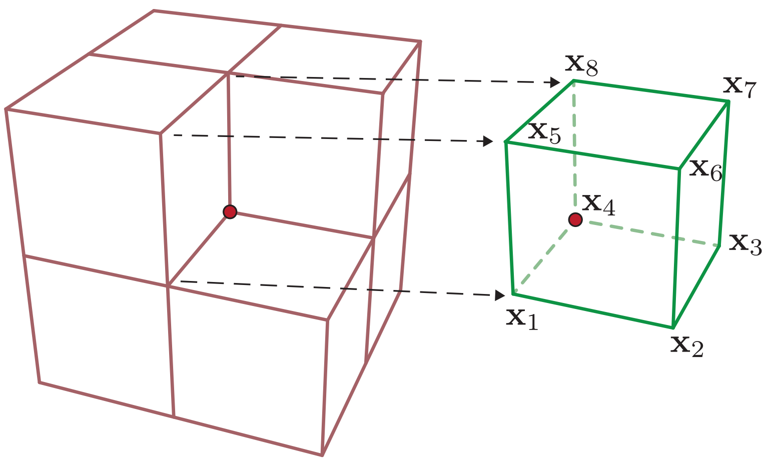
Regularized shape distortion measure

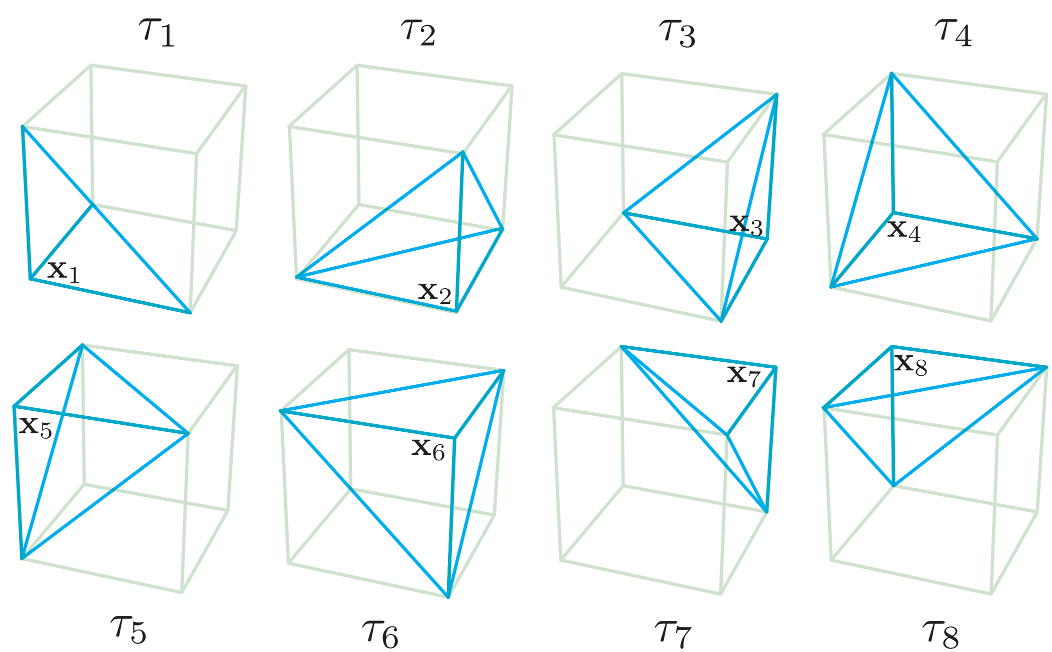
$$\eta^*(S) = \frac{\|S\|^2}{2h(\sigma)}$$

Distortion of a quadrilateral

$$\eta^*(\mathbf{x}) = \left[\frac{1}{4}\sum_{i=1}^{4}(\eta^*(S_i))^p\right]^{1/p}$$

Objective function for hexahedra





Objective Function

$$\boxed{K^*(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N}(\eta_i^*(\mathbf{x}))^p}$$

Distortion of a hexahedron

$$\eta^*(\mathbf{x}) = \left[\frac{1}{8}\sum_{i=1}^{8}(\eta^*(S_i))^p\right]^{1/p}$$

Regularized shape distortion measure of a tetrahedron

$$\eta^*(S) = \frac{\|S\|^2}{3h(\sigma)^{2/3}}$$

Hexahedral mesh optimization

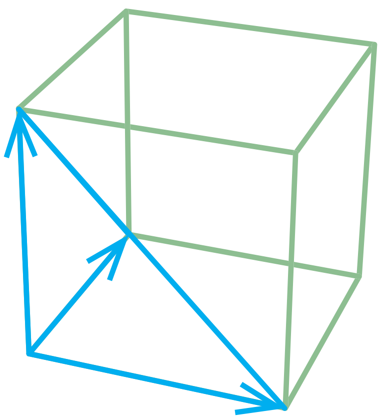
- **Optimization** can be carried out **by decomposing** each element **into simplices** (triangles or tetrahedra).
- **Objective function** is **based on** the shape **distortion** measure of a **simplex.**
- **For quadrilaterals**, optimization **guarantee the validity** of the elements (Validity of 4 triangles is a necessary and sufficient condition for validity of the quadrilateral).
- **For hexahedra**, optimization **does not guarantee the validity** of the elements (Validity of 8 tetrahedra is a necessary but not sufficient condition for validity of a hexahedra).

What other conditions may be used?

Sets of tetrahedra

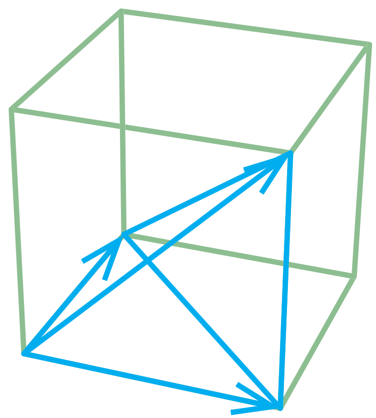
A hexahedron can be subdivided into 58 tetrahedra

The tetrahedra can be grouped in four groups:



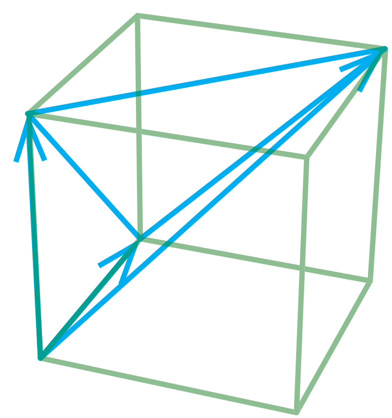
α

8 tets



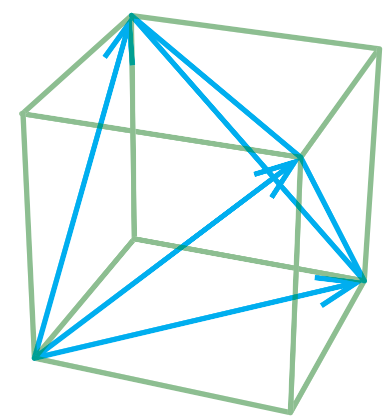
β

24 tets



γ

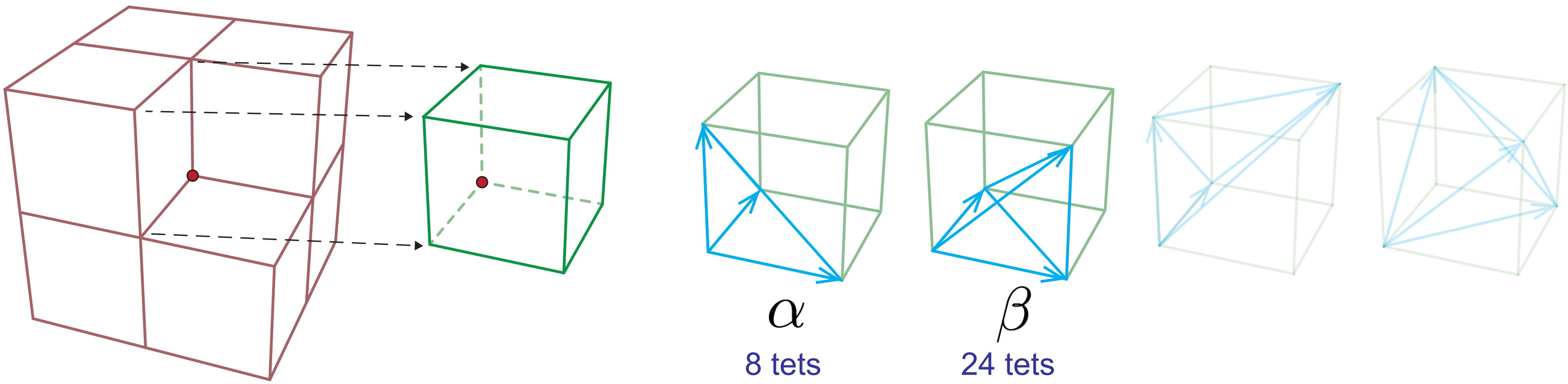
24 tets



κ

2 tets

Objective functions for hexahedra



Objective Function

$$K^*(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N}(\eta_i^*(\mathbf{x}))^p$$

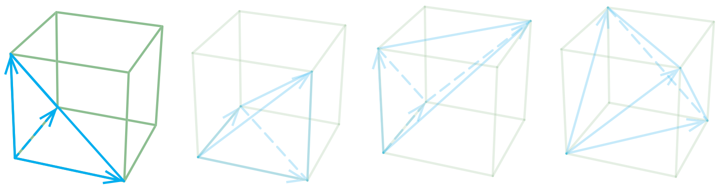
Distortion of a hexahedron

$$\eta^*(\mathbf{x}) = \left[\frac{1}{32}\sum_{i=1}^{32}(\eta^*(S_i))^p\right]^{1/p}$$

Regularized shape distortion measure of a tetrahedron

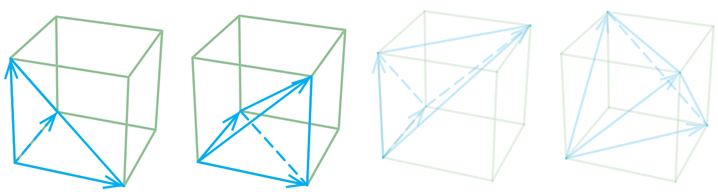
$$\eta^*(S) = \frac{\|S\|^2}{3h(\sigma)^{2/3}}$$

Objective functions for hexahedra



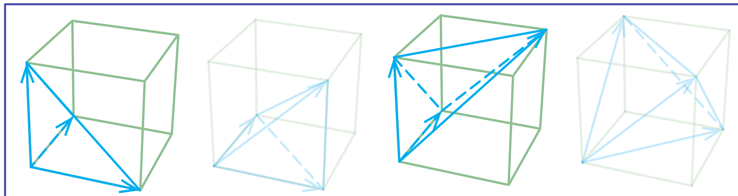
α

Necessary but not sufficient conditions



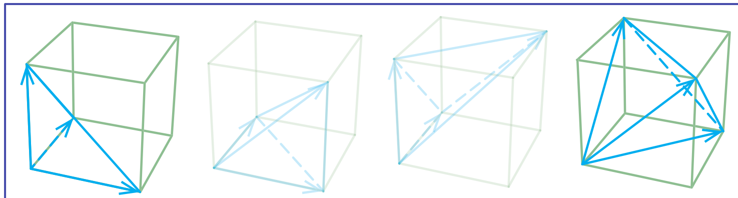
$\alpha \quad \beta$

Neither necessary nor sufficient conditions



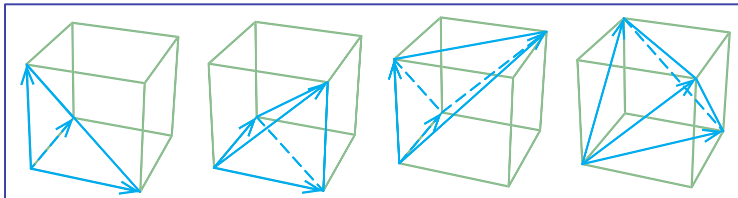
$\alpha \quad \gamma$

Neither necessary nor sufficient conditions



$\alpha \quad \kappa$

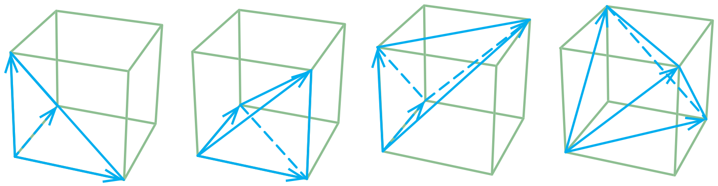
Neither necessary nor sufficient conditions



$\alpha \quad \beta \quad \gamma \quad \kappa$

Not necessary but **sufficient** conditions

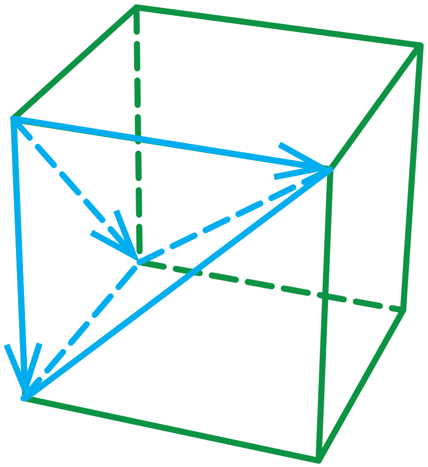
Objective functions for hexahedra



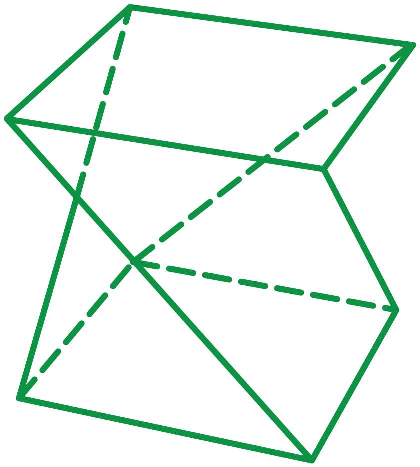
$\alpha \quad \beta \quad \gamma \quad \kappa$

Not necessary but **sufficient** conditions

Only 30% of valid elements satisfy this

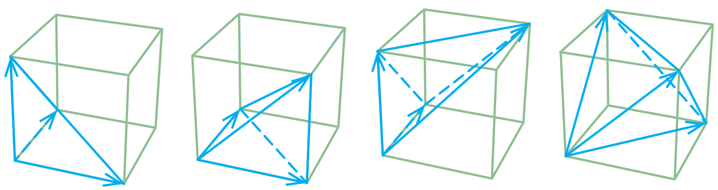


rotate top cover -90º



$$J(\boldsymbol{\xi}) > 0,\ \boldsymbol{\xi} \in [-1, 1]^3$$

Objective functions for hexahedra

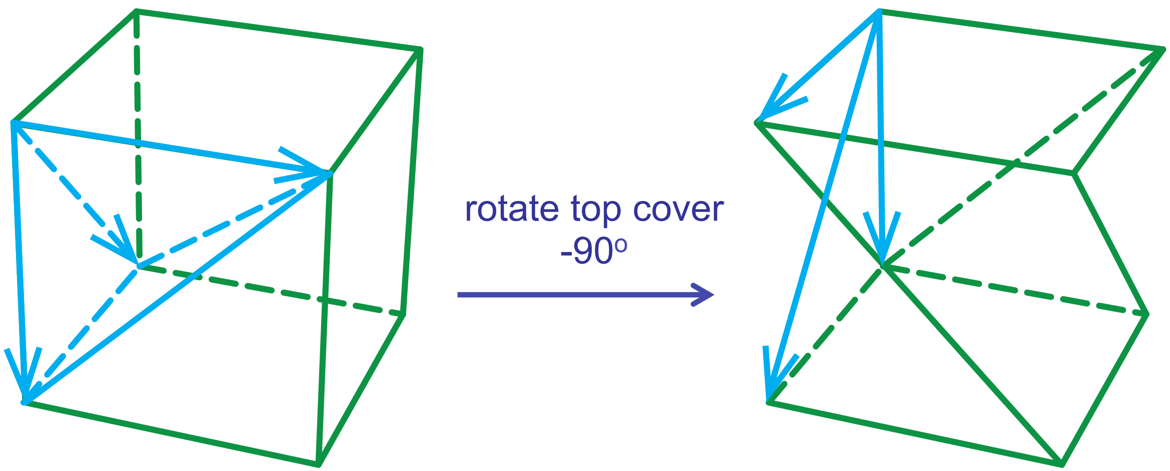


$\alpha \quad \beta \quad \gamma \quad \kappa$

Not necessary but **sufficient** conditions

Only 30% of valid elements satisfy this

Very restrictive conditions !

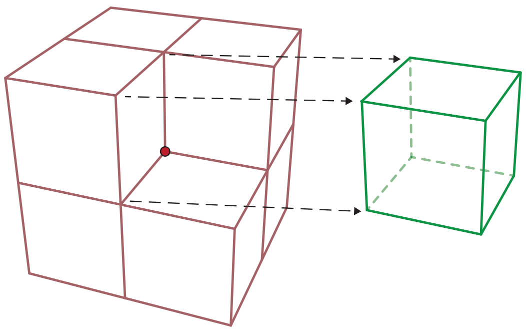


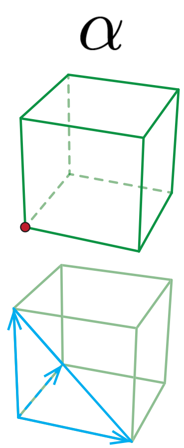
Becomes into a flat tetrahedron

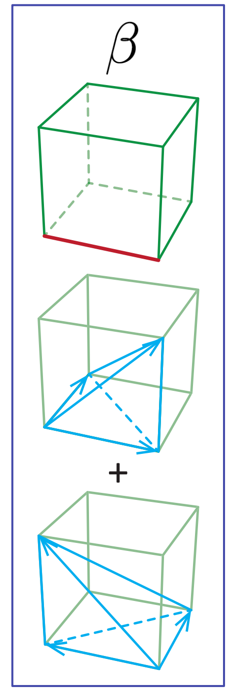
$$J(\boldsymbol{\xi}) > 0,\ \boldsymbol{\xi} \in [-1,1]^3$$

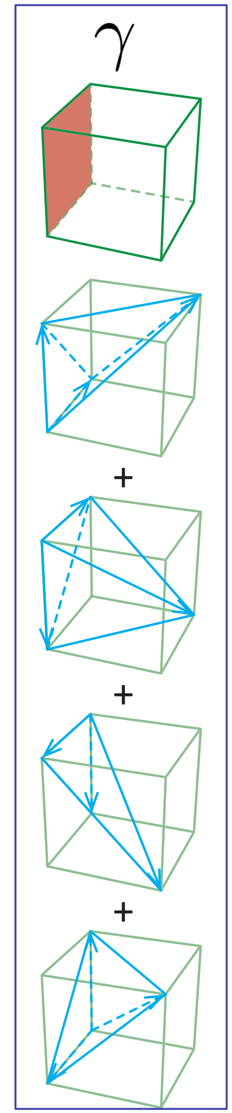
$$\boxed{\sigma(\tau) = 0}$$

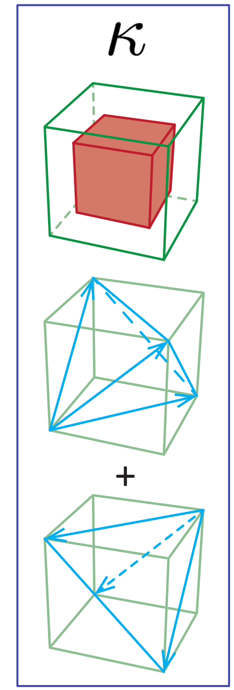
Objective functions for hexahedra











66% of valid elements satisfy this

Less restrictive **sufficient** conditions

O. V. Ushakova, **Nondegeneracy tests for hexahedral cells**, Computer Methods in Applied Mechanics and Engineering 200 (17-20) (2011) 1649–1658.

Based on sum of volumes

Experiment

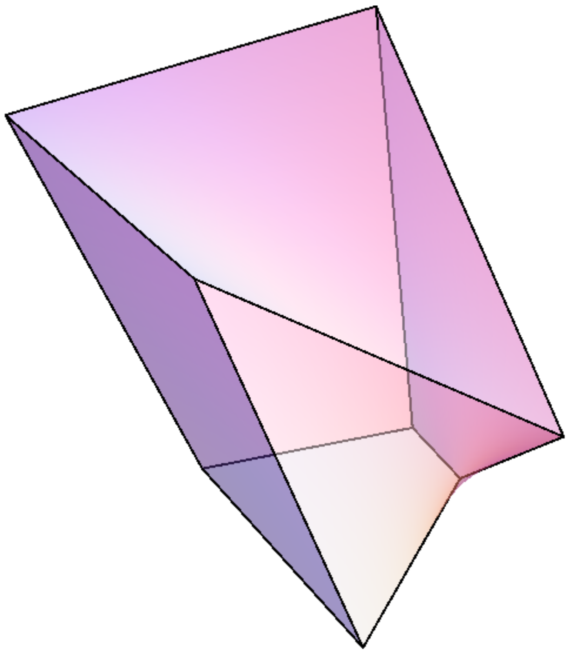
Comparative study

Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers,
DOI 10.1007/s00366-016-0454-1

Goal: compare the untangling capability of different objective functions

First step

Generate high distorted elements with random vertices



Valid element → always exists feasible region for all vertices

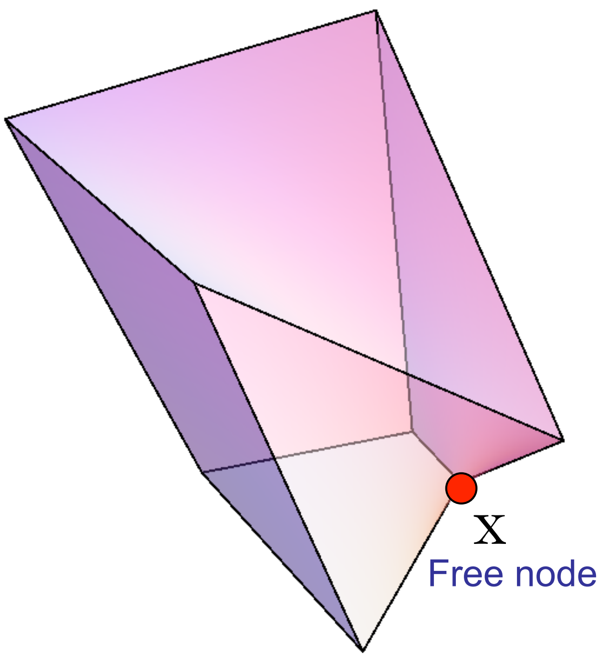
Comparative study

Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers,
DOI 10.1007/s00366-016-0454-1

Goal: compare the untangling capability of different objective functions

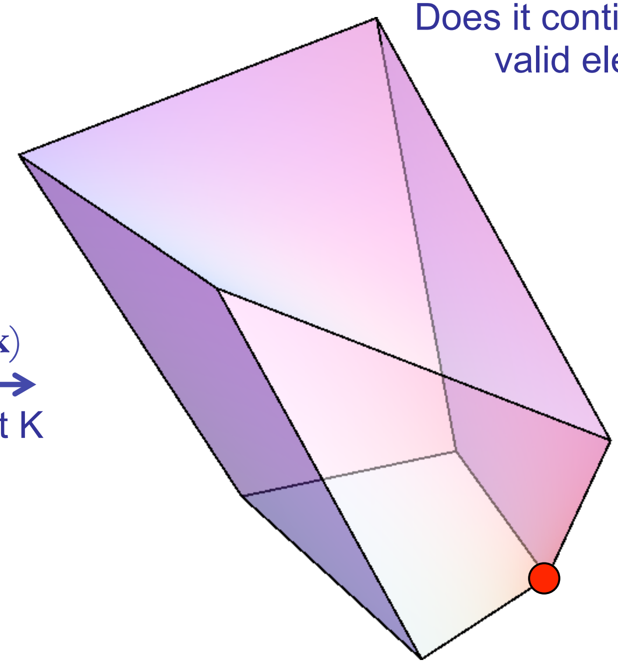
Second step

Optimize the element by moving one vertex



$$\mathbf{x} = \min K(\mathbf{x})$$

Using different K

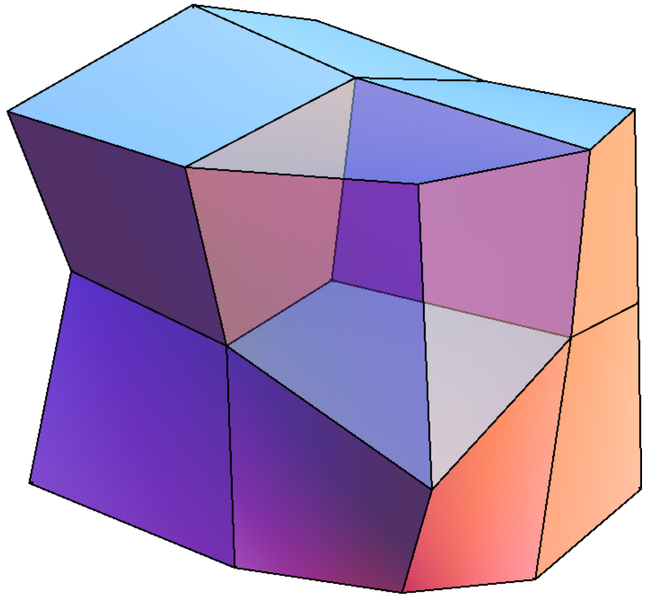


Does it continue being a valid element?

Comparative study

Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers,
DOI 10.1007/s00366-016-0454-1

Same experiment with local meshes

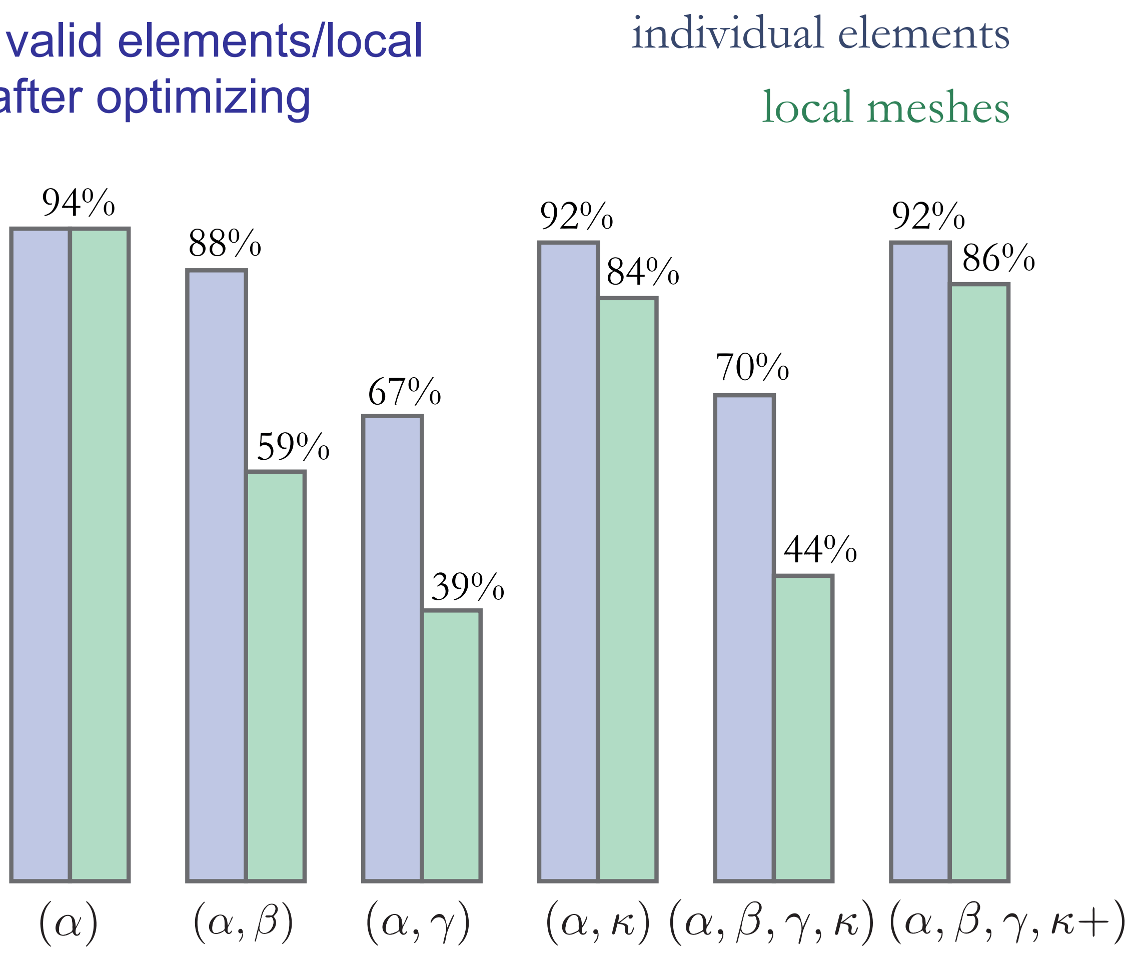


- Valid and hight distorted local mesh
- Optimize central node
- Valid local mesh after optimizing?

Comparative study

Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers, DOI 10.1007/s00366-016-0454-1

Percentage of valid elements/local meshes after optimizing



Comparative study

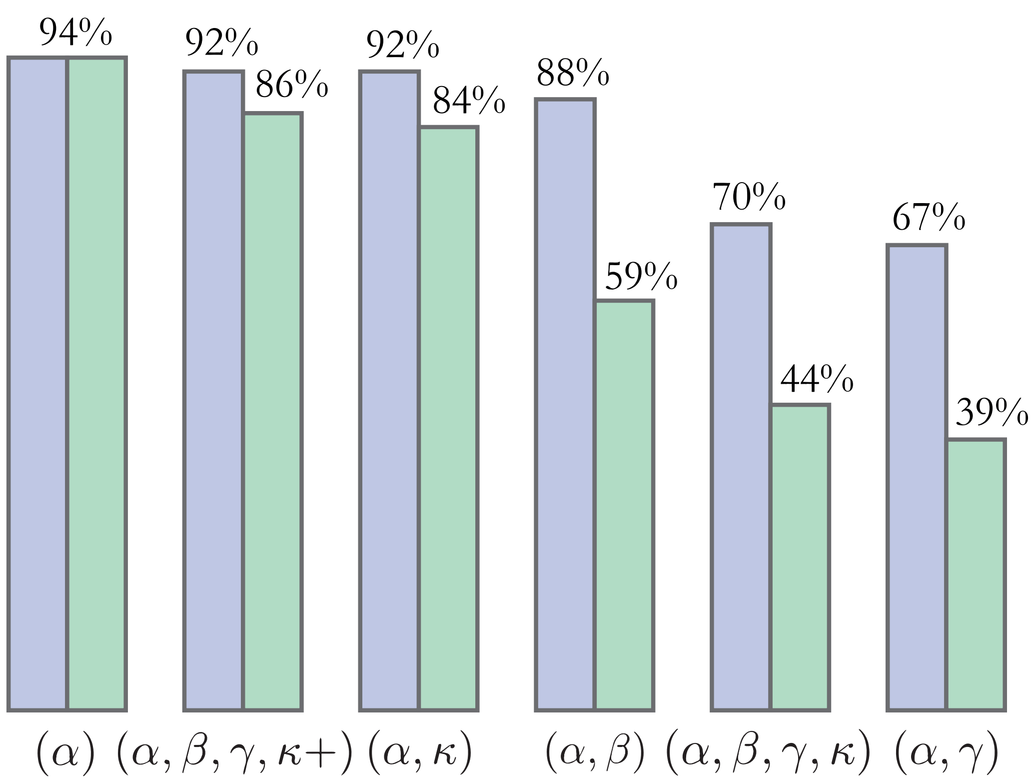
Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers, DOI 10.1007/s00366-016-0454-1

SIANI

Percentage of valid elements/local meshes after optimizing

individual elements

local meshes



How optimize without simplex decomposition?

Objective functions for hexahedra

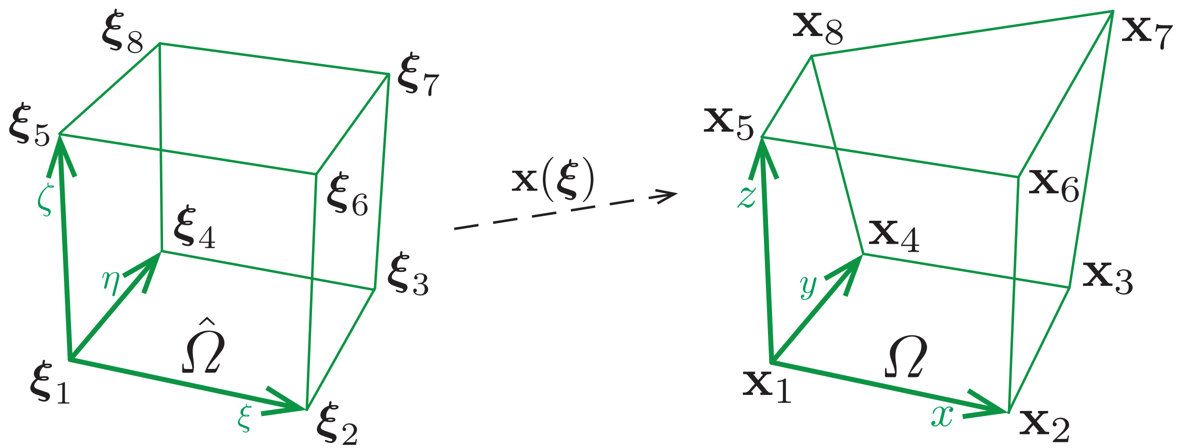
Using the weighted Jacobian matrix of the trilinear transformation

$$S(\boldsymbol{\xi}) = \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right)$$

$$\begin{aligned}
\boldsymbol{\xi}_1 &= (-1,-1,-1) \\
\boldsymbol{\xi}_2 &= (1,-1,-1) \\
\boldsymbol{\xi}_3 &= (1,1,-1) \\
\boldsymbol{\xi}_4 &= (-1,1,-1) \\
\boldsymbol{\xi}_5 &= (-1,-1,1) \\
\boldsymbol{\xi}_6 &= (1,-1,1) \\
\boldsymbol{\xi}_7 &= (1,1,1) \\
\boldsymbol{\xi}_8 &= (-1,1,1)
\end{aligned}$$

Reference element

Physical element



Pointwise distortion

$$\eta^*(\boldsymbol{\xi}) = \frac{\|S(\boldsymbol{\xi})\|^2}{3h(\sigma(\boldsymbol{\xi}))^{2/3}}$$

$$\boldsymbol{\xi} \in [-1,1]^3$$

Global distortion

$$\eta^*_{\Omega} = \frac{1}{V_{\hat{\Omega}}} \int_{\hat{\Omega}} \eta^*(\boldsymbol{\xi})\, d\hat{\Omega}$$

Objective functions for hexahedra

Using the weighted Jacobian matrix of the trilinear transformation

$$S(\boldsymbol{\xi}) = \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right)$$

$\boldsymbol{\xi}_1 = (-1,-1,-1)$
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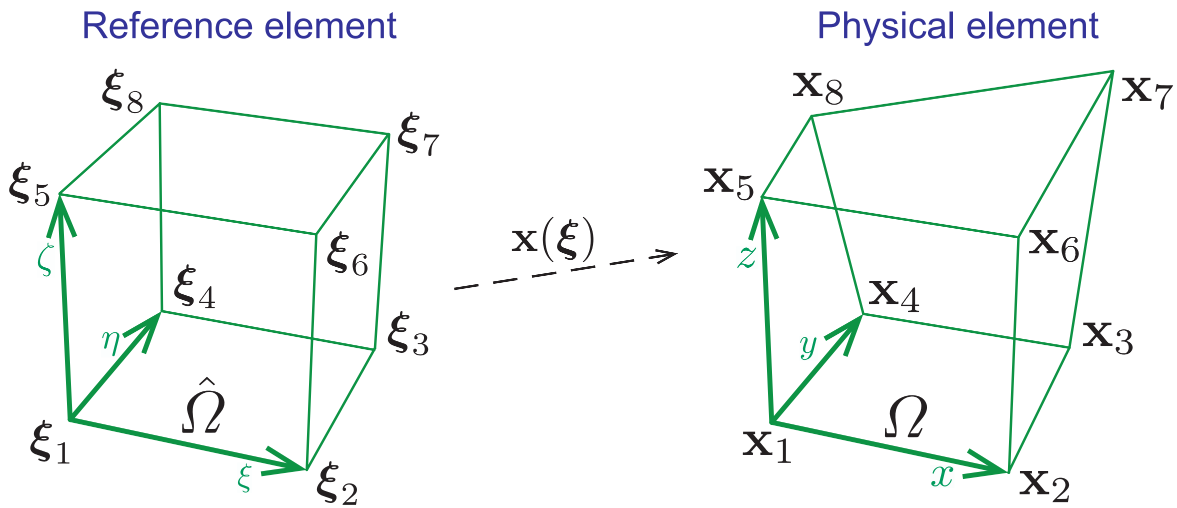
$\boldsymbol{\xi}_4 = (-1,1,-1)$
$\boldsymbol{\xi}_5 = (-1,-1,1)$
$\boldsymbol{\xi}_6 = (1,-1,1)$
$\boldsymbol{\xi}_7 = (1,1,1)$
$\boldsymbol{\xi}_8 = (-1,1,1)$



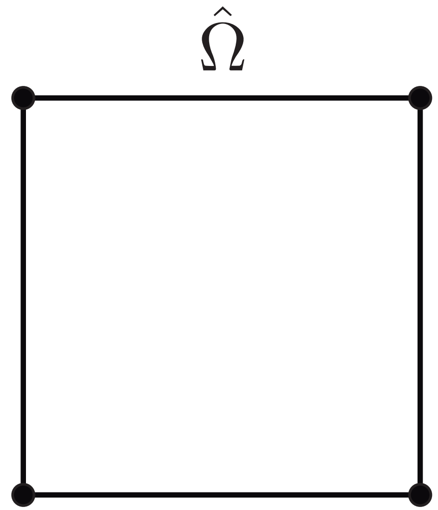
Objective function

$$\boxed{K^*(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M} \eta^*_{\Omega_m}(\mathbf{x})}$$

Numerical quadrature of the distortion

$$\int_{\hat{\Omega}} \eta^*(\boldsymbol{\xi})\, d\hat{\Omega} \approx \sum_{i,j,k=1}^{2} w_{ijk}\, \eta^*(\boldsymbol{\xi}_{ijk})$$

quadrature points

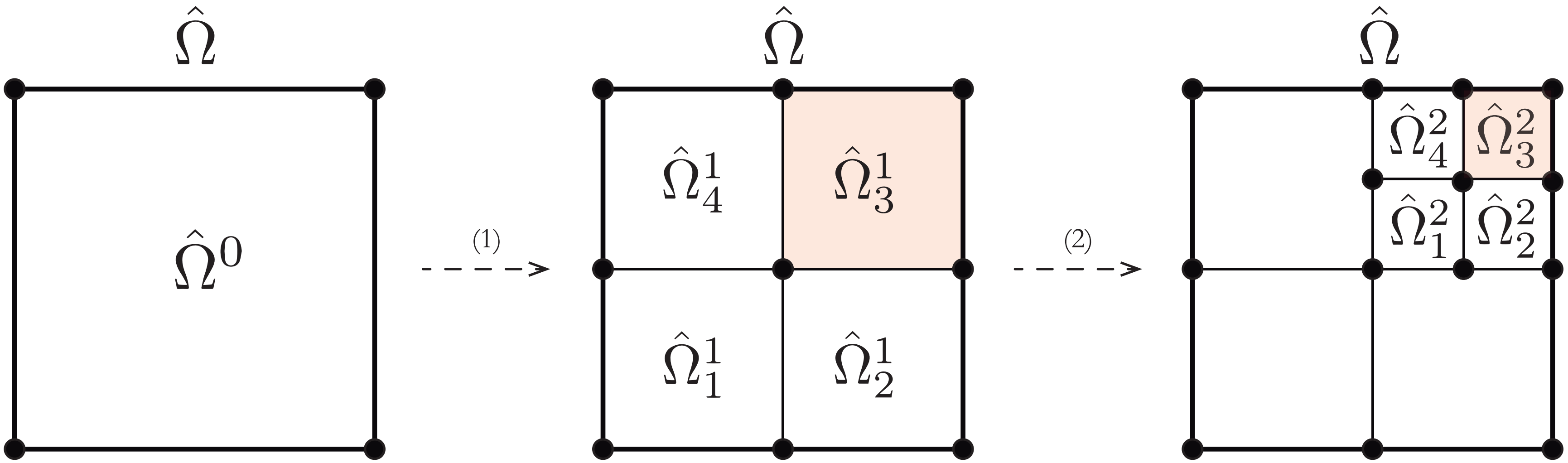


Objective functions for hexahedra

Numerical quadrature of the distortion

$$\int_{\hat{\Omega}} \eta^*(\boldsymbol{\xi})\, d\hat{\Omega} \approx \sum_{i,j,k=1}^{2} w_{ijk}\, \eta^*(\boldsymbol{\xi}_{ijk})$$

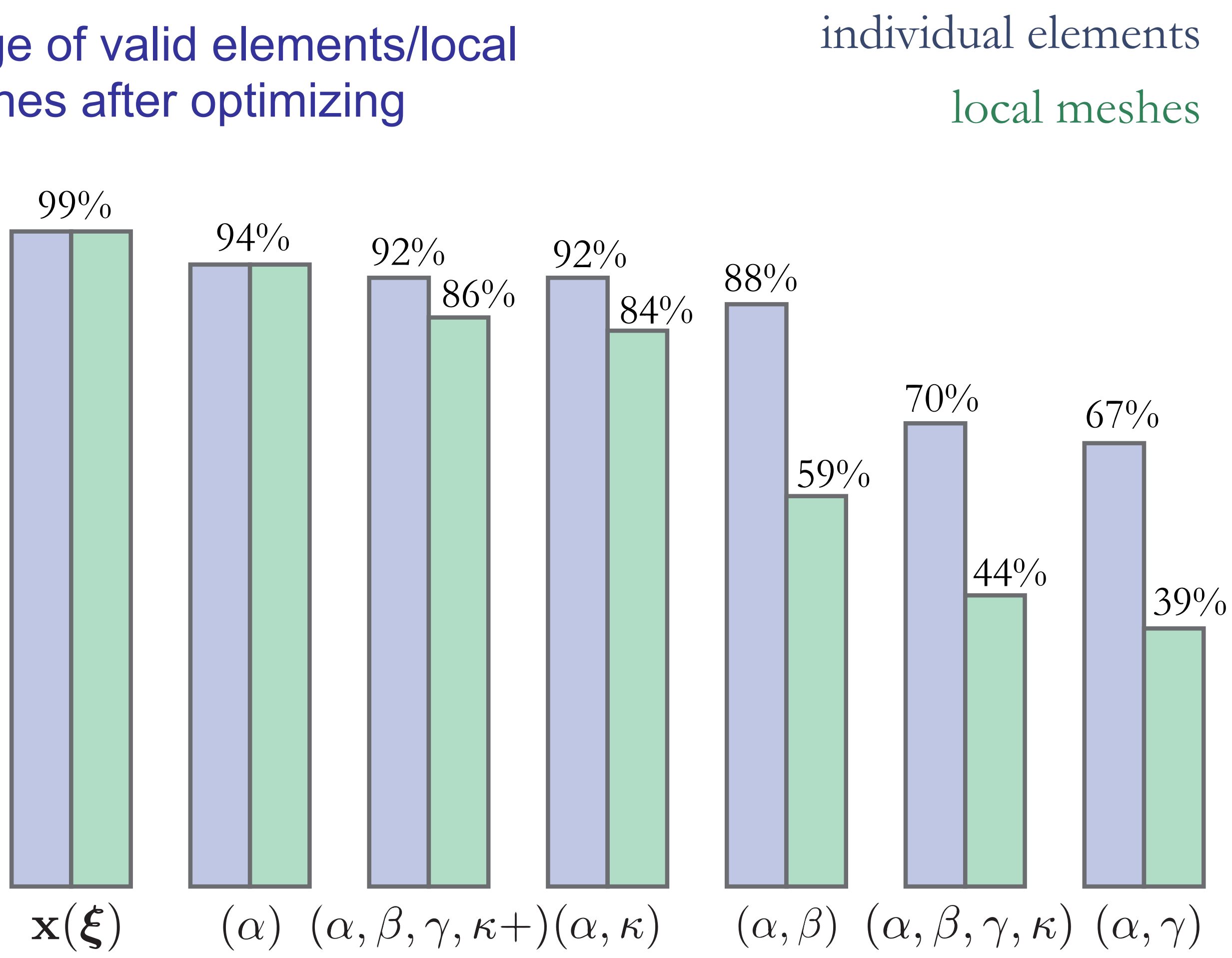
adaptive distribution of quadrature points



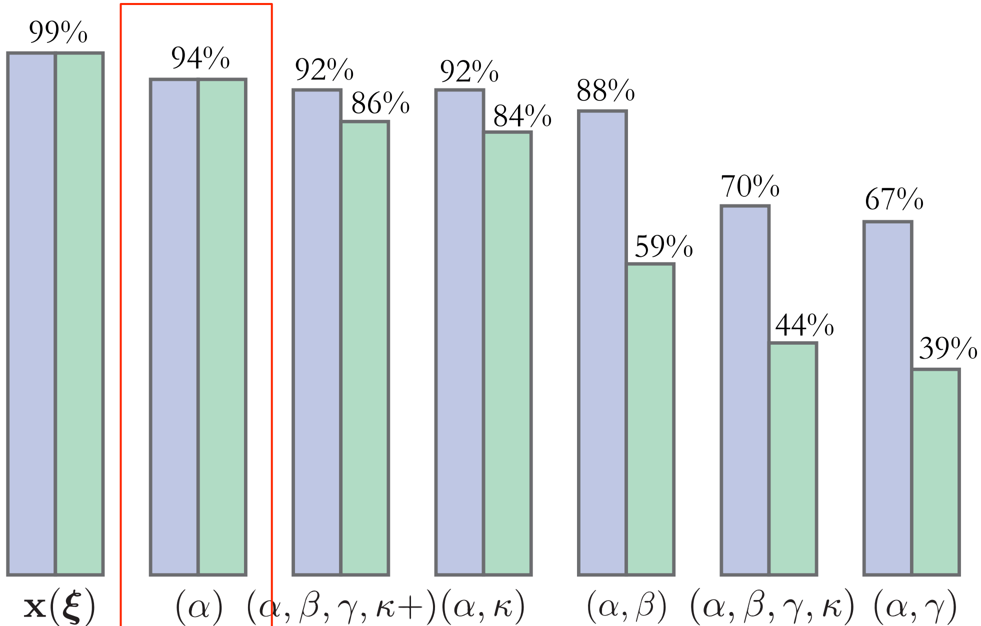
Comparative study

Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers,
DOI 10.1007/s00366-016-0454-1

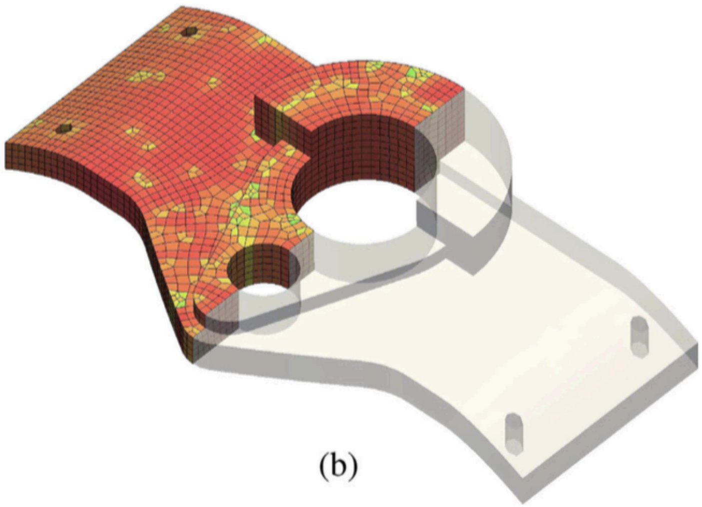
Percentage of valid elements/local meshes after optimizing



Conclusions



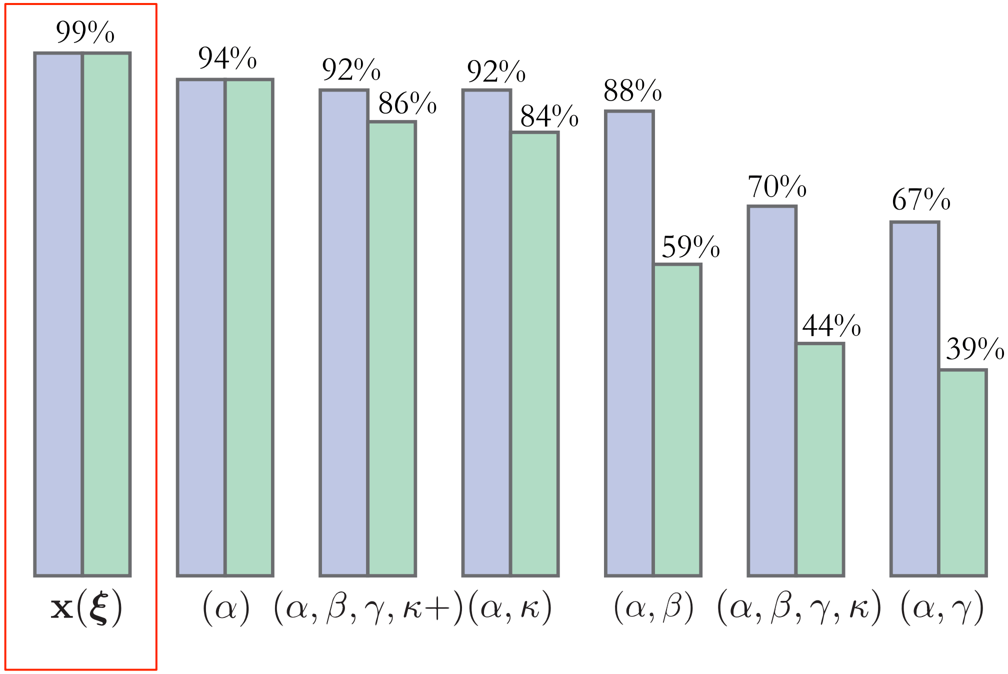
In real meshes:



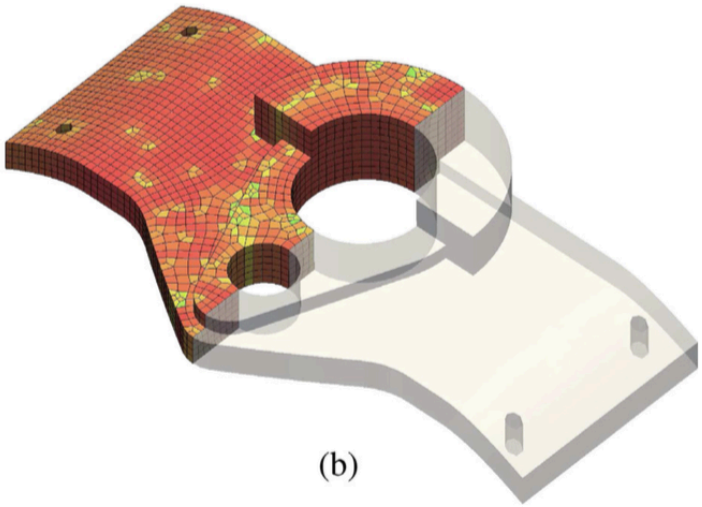
(b)

Good results and less computationally expensive than others

Conclusions



In real meshes:



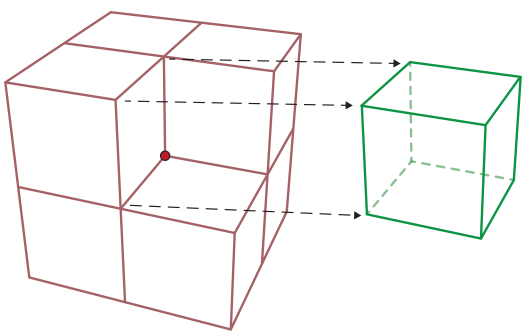
(b)

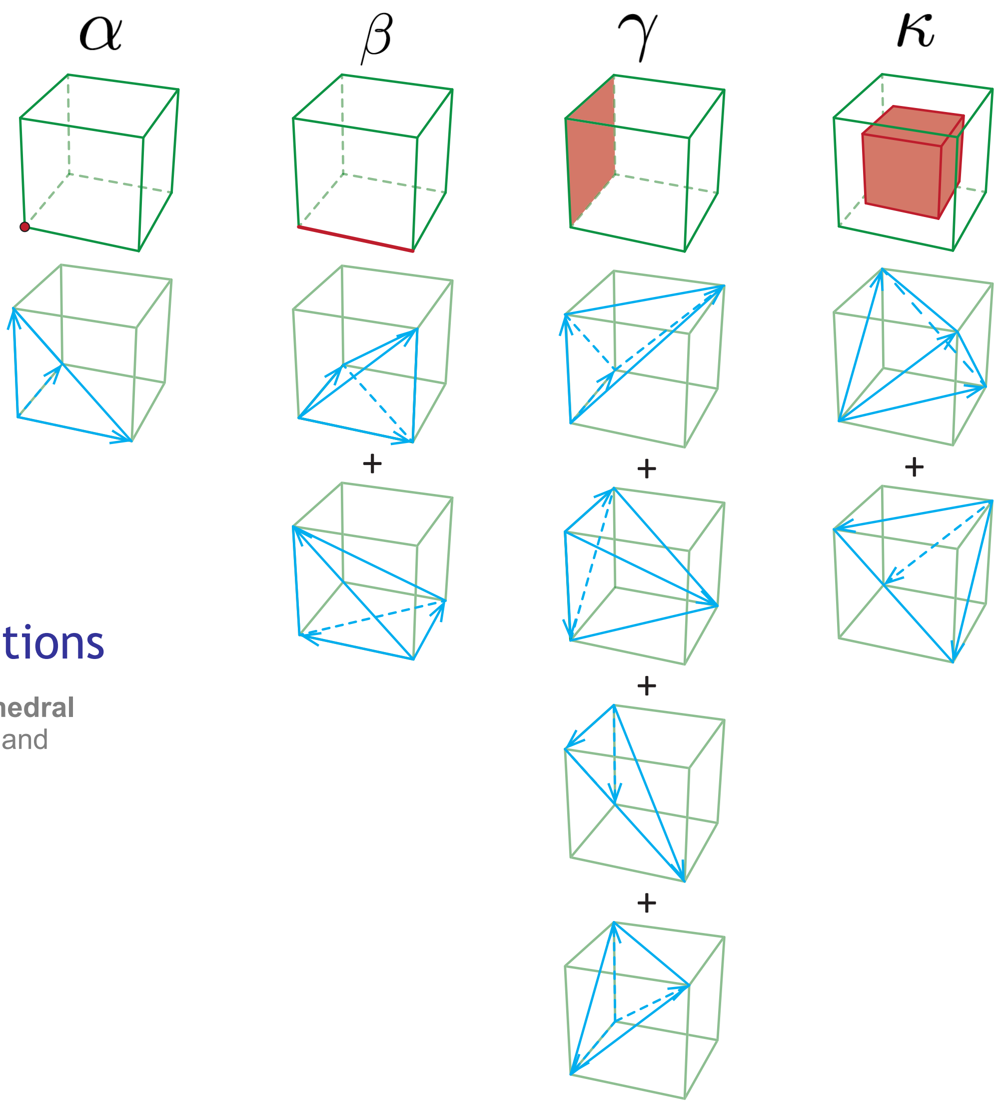
Use in the last steps of optimization

Thanks for you attention

Any questions?

Objective functions for hexahedra

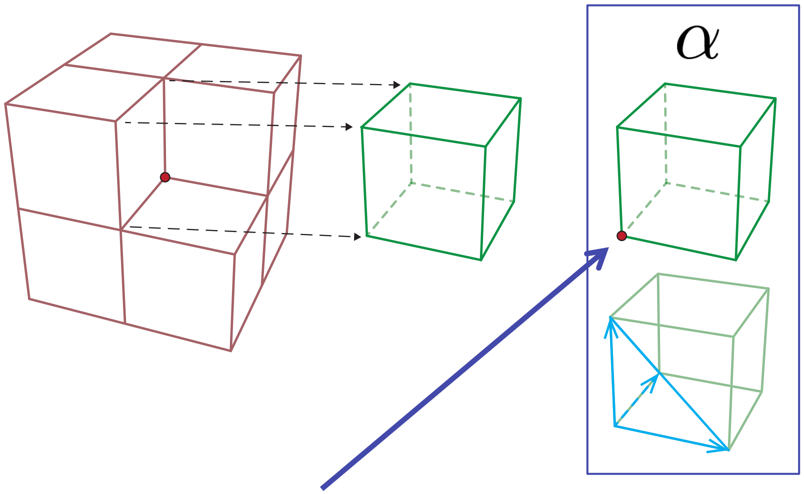




Less restrictive sufficient conditions

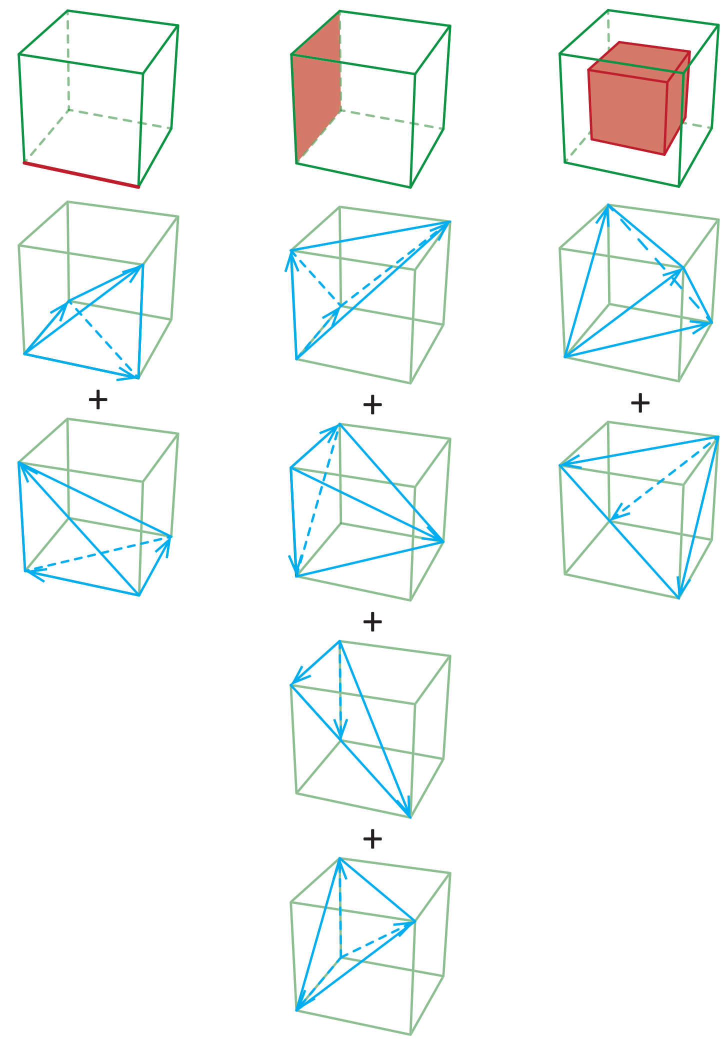
O. V. Ushakova, **Nondegeneracy tests for hexahedral cells**, Computer Methods in Applied Mechanics and Engineering 200 (17-20) (2011) 1649–1658.

Objective functions for hexahedra

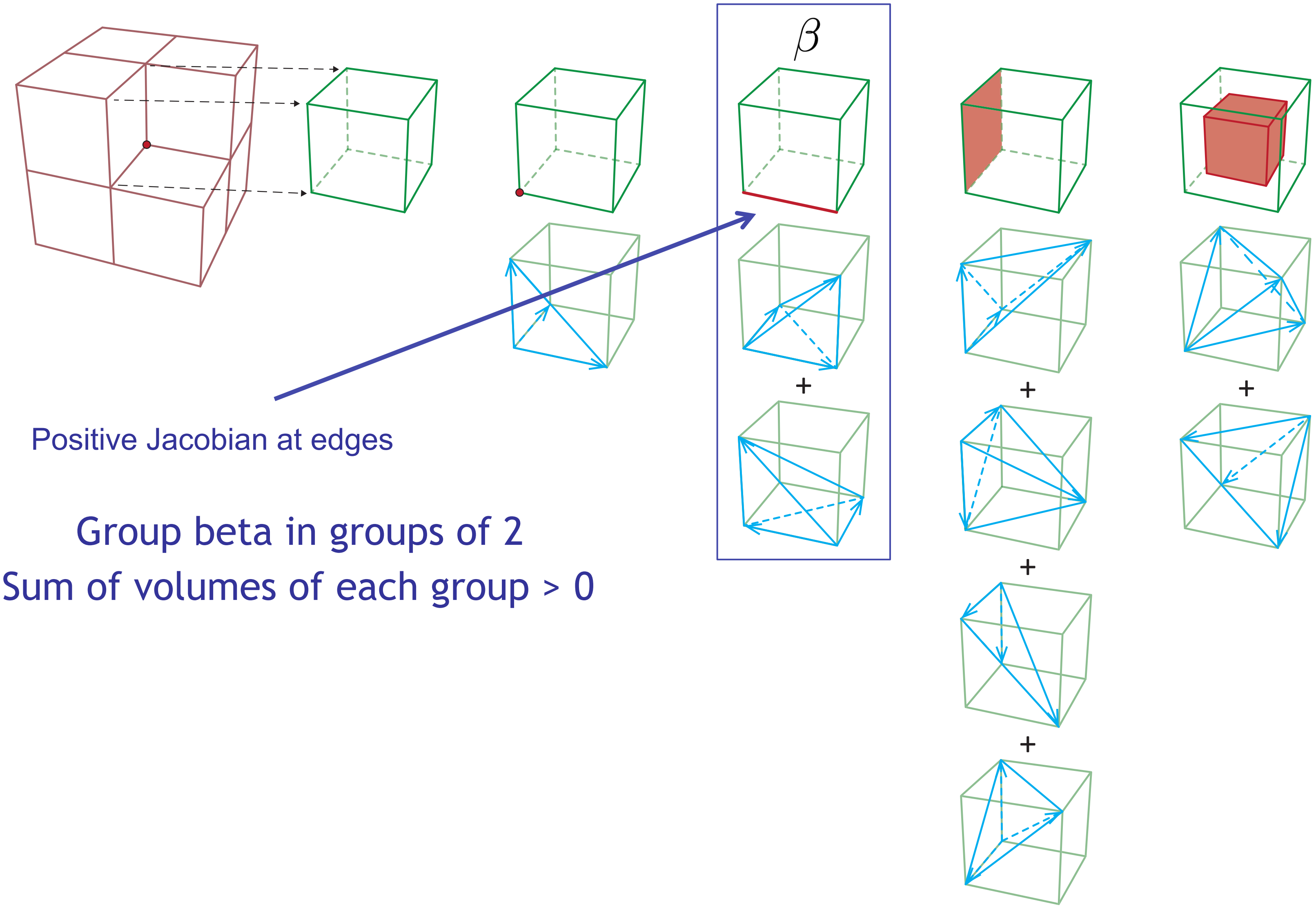


Positive Jacobian at vertices

Validity of each alpha tetrahedron



Objective functions for hexahedra

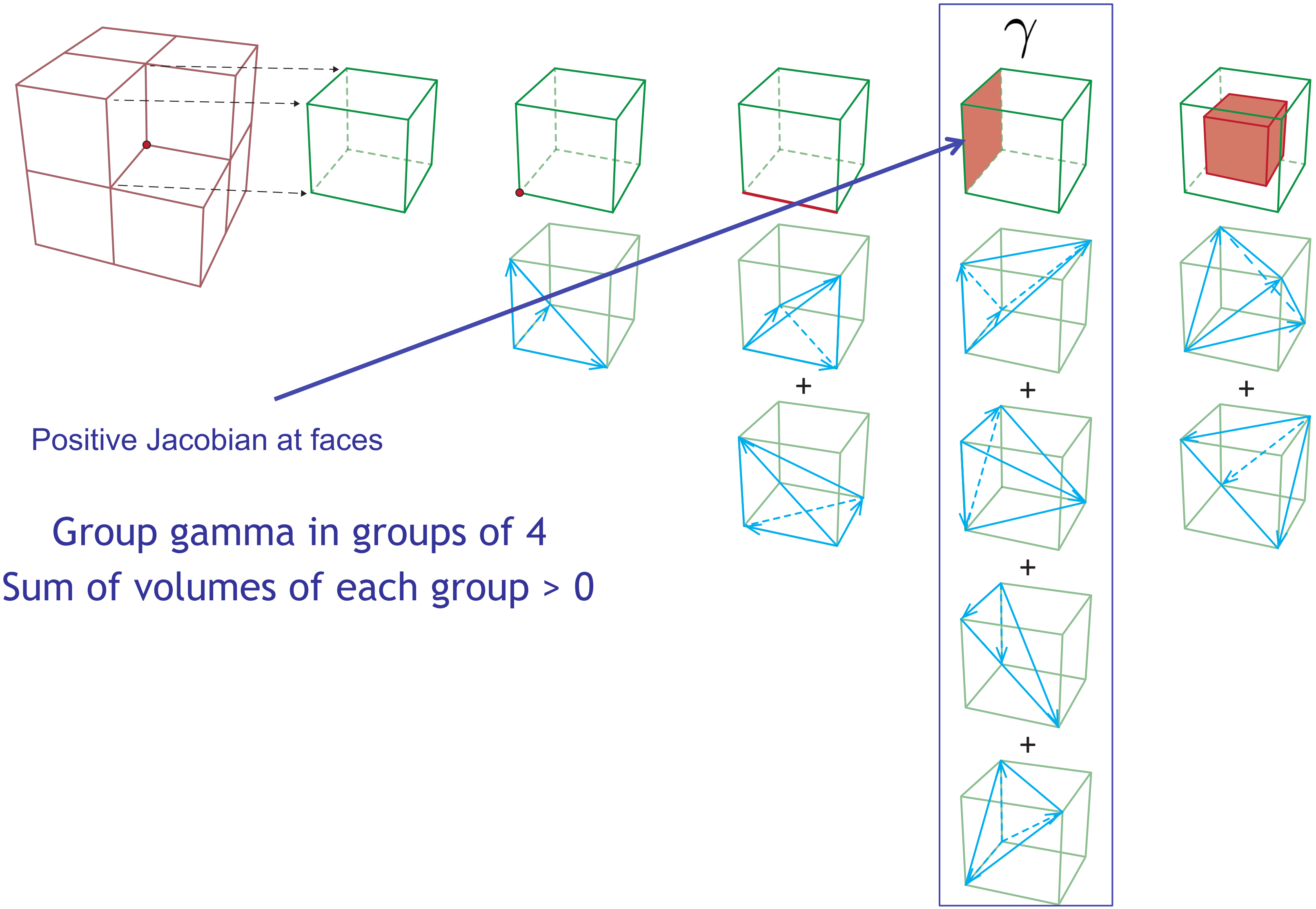


Positive Jacobian at edges

Group beta in groups of 2

Sum of volumes of each group > 0

Objective functions for hexahedra

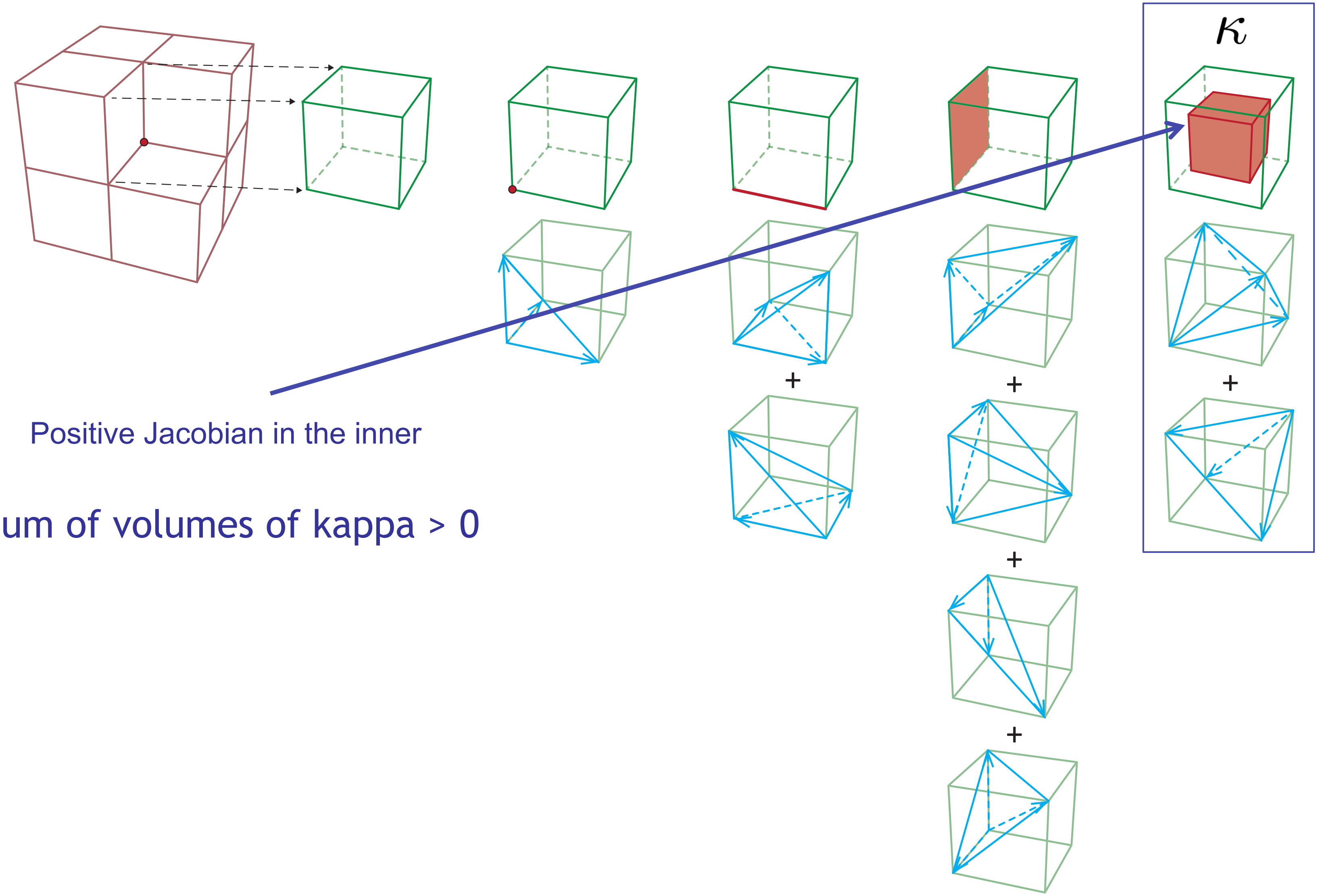


Positive Jacobian at faces

Group gamma in groups of 4

Sum of volumes of each group > 0

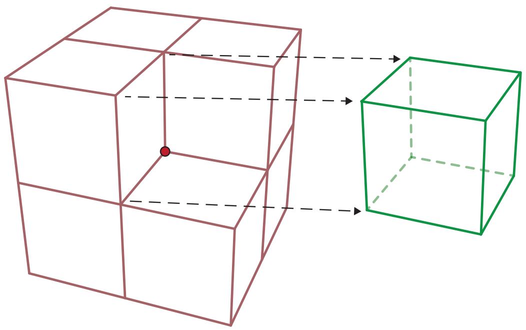
Objective functions for hexahedra

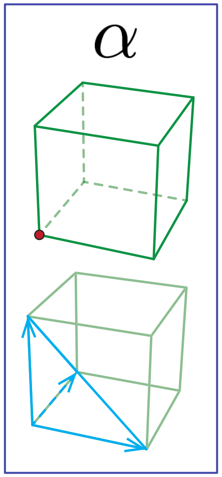


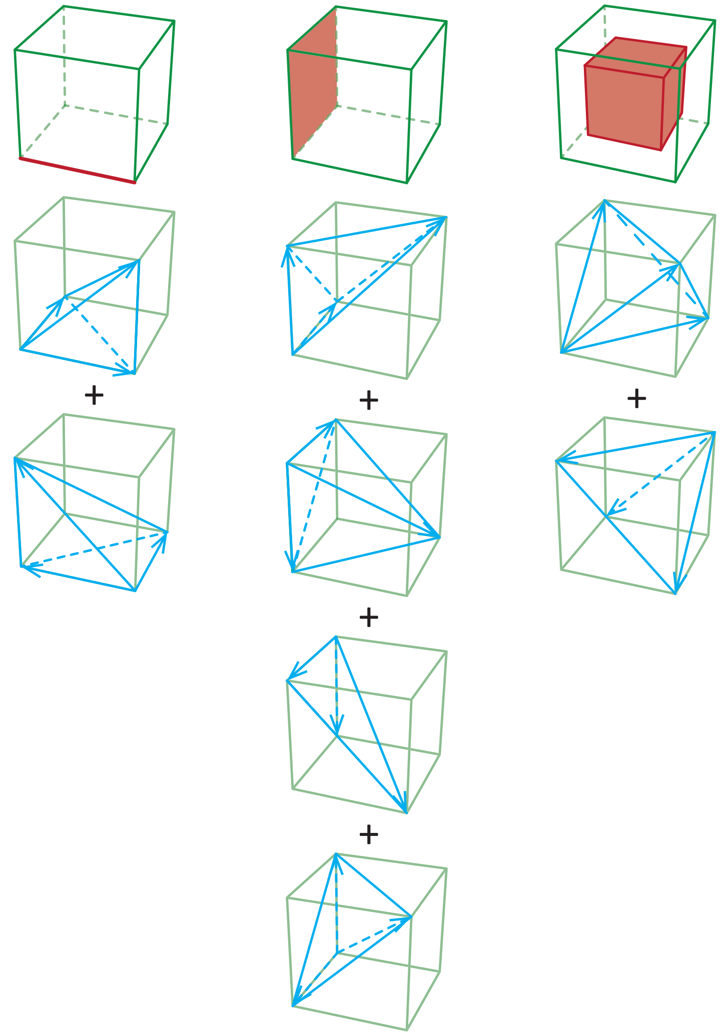
Positive Jacobian in the inner

Sum of volumes of kappa > 0

Objective functions for hexahedra





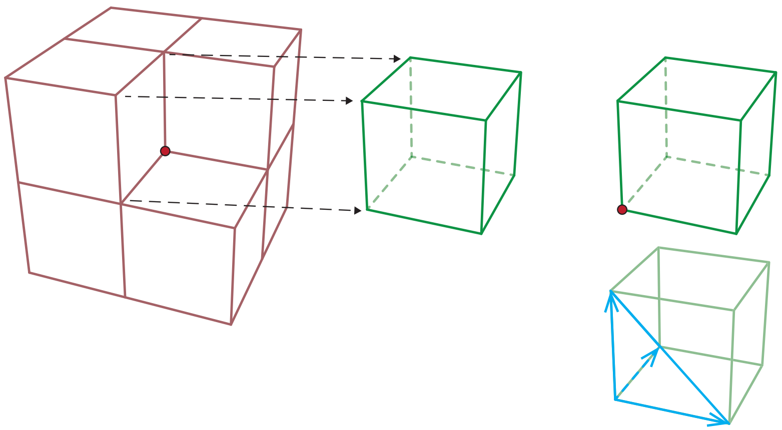


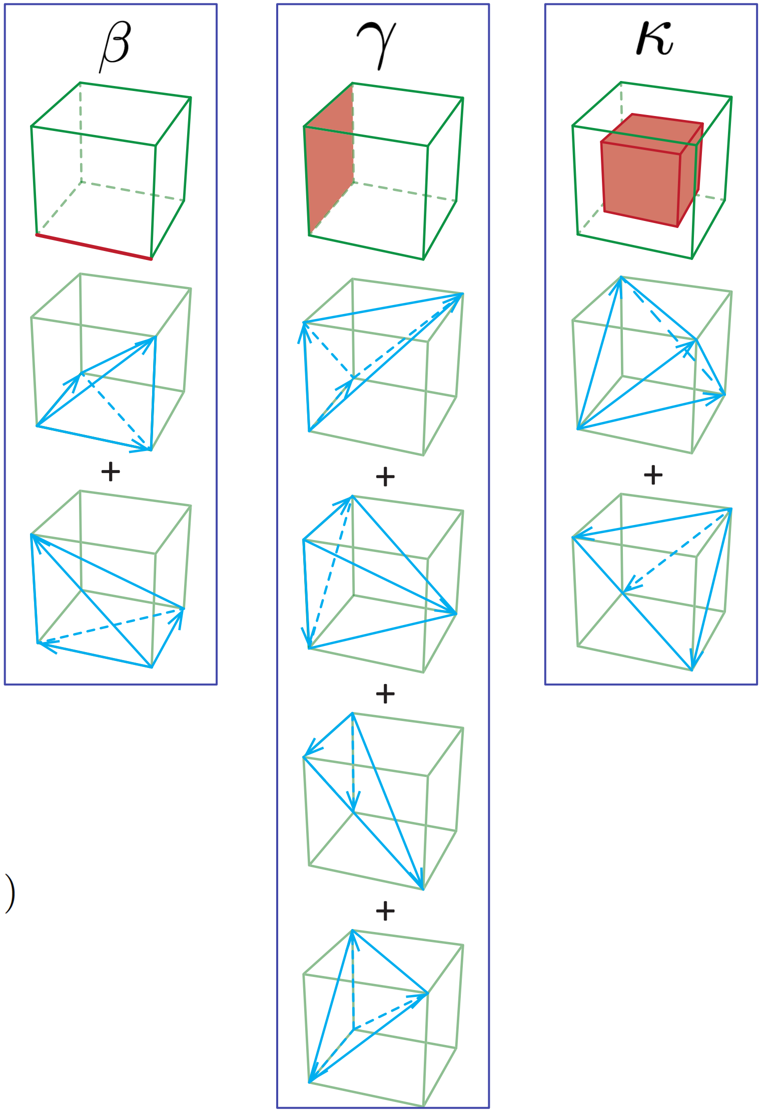
Validity of each alpha tetrahedron

$$\eta^*(S) = \frac{\|S\|^2}{3h(\sigma)^{2/3}}$$

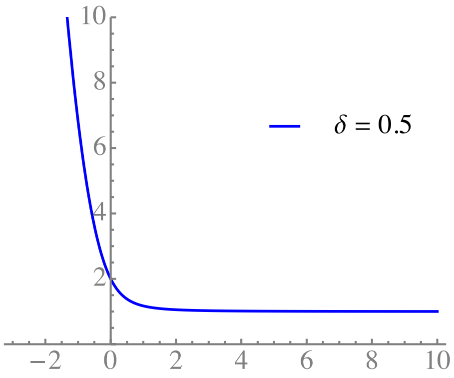
Imposes validity and shape

Objective functions for hexahedra





Sum of volumes of each group > 0



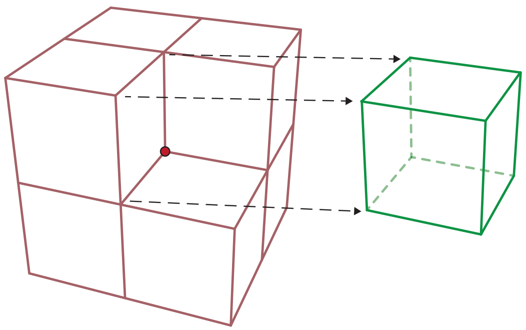
$$h'(\sigma)^{-1} = 2 - \frac{\sigma}{h(\sigma)}$$

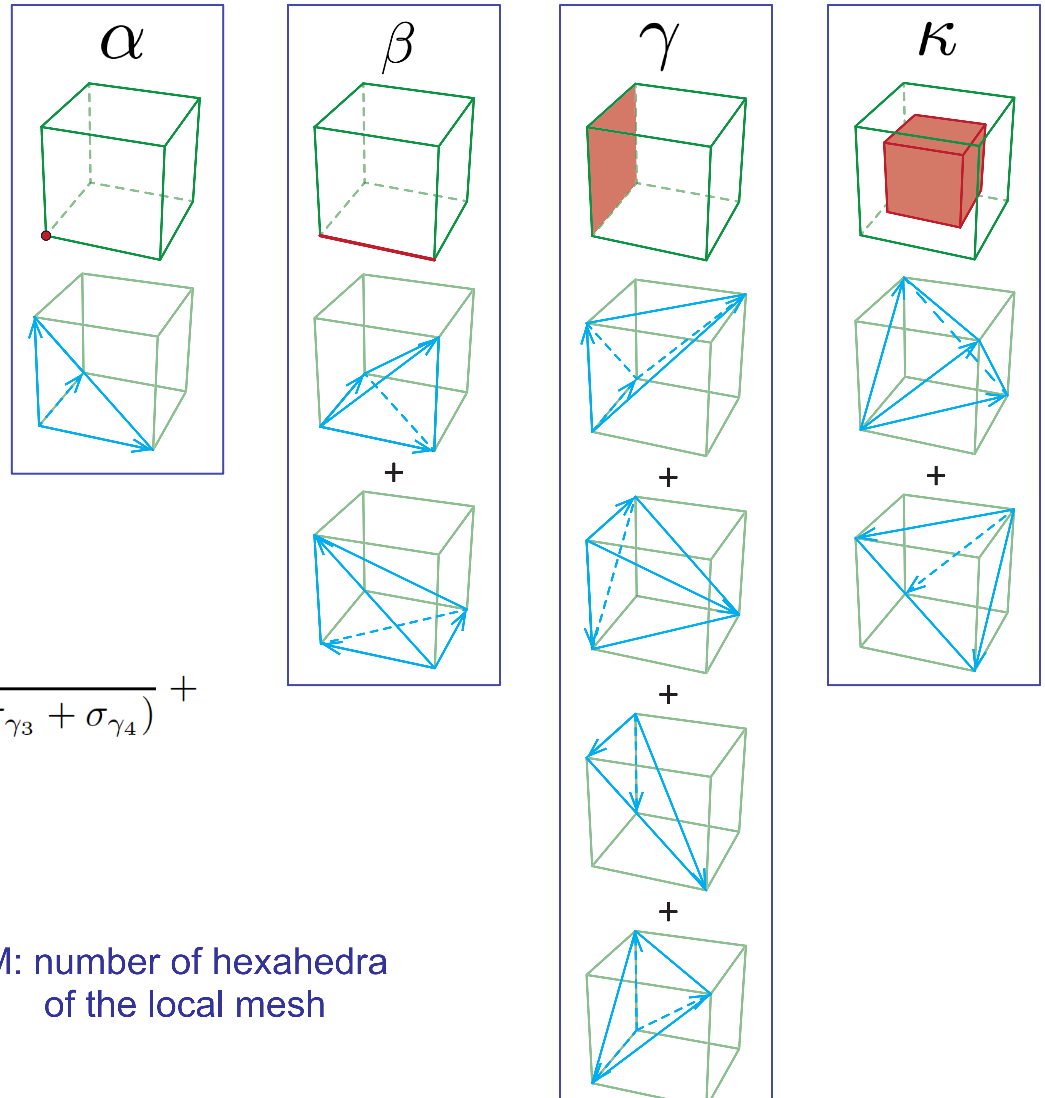
i.e. $\quad \sigma = \sigma(\beta_1) + \sigma(\beta_2)$

Imposes only barriers

Objective functions for hexahedra

Strategies for optimization of hexahedral meshes and their comparative study, Engineering With Computers,
DOI 10.1007/s00366-016-0454-1





Objective function
(simplified)

$$k^* = \sum \eta^*(S_{\alpha_i}) + \sum \frac{1}{h'(\sigma_{\beta_1} + \sigma_{\beta_2})} + \\ + \sum \frac{1}{h'(\sigma_{\gamma_1} + \sigma_{\gamma_2} + \sigma_{\gamma_3} + \sigma_{\gamma_4})} + \\ + \frac{1}{h'(\sigma_{\kappa_1} + \sigma_{\kappa_2})}$$

$$K^*(\mathbf{x}) = \sum_{m=1}^{M} (k_m^*)^p(\mathbf{x})$$

M: number of hexahedra
of the local mesh