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Normalized Frobenius condition number of the orthogonal projections of the identity ☆

Antonio Suárez, Luis González*

Department of Mathematics, University of Las Palmas de Gran Canaria, 35017 Las Palmas de Gran Canaria, Spain

Abstract

This paper deals with the orthogonal projection (in the Frobenius sense) AN of the identity matrix I onto the matrix subspace AS ($A \in \mathbb{R}^{n\times n}$, S being an arbitrary subspace of $\mathbb{R}^{n\times n}$). Lower and upper bounds on the normalized Frobenius condition number of matrix AN are given. Furthermore, for every matrix subspace $S \subset \mathbb{R}^{n\times n}$, a new index $\widehat{\kappa}_F(A,S)$, which generalizes the normalized Frobenius condition number of matrix A, is defined and analyzed.

Keywords: Frobenius norm minimization; Orthogonal projections; Frobenius condition number
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1. Introduction

Throughout this paper, A^T and $tr(A)$ denote, as usual, the transpose and the trace of matrix $A \in \mathbb{R}^{n\times n}$, while I denotes the identity matrix of order n. Let $\langle \cdot,\cdot \rangle_F$ and $\|\cdot\|_F$ denote the Frobenius inner product and matrix norm, defined on the matrix space $\mathbb{R}^{n\times n}$. In the following, the terms orthogonality, angle and cosine will be used in the sense of the Frobenius inner product. The symbols $\kappa_F(\cdot)$ and $\widehat{\kappa}_F(\cdot)$ stand for the classical and for the normalized Frobenius condition numbers, respectively, i.e., for every nonsingular $n\times n$ real matrix M

$$\kappa_F(M) = \|M\|_F \left\|M^{-1}\right\|_F, \quad \widehat{\kappa}_F(M) = \frac{1}{n}\|M\|_F \left\|M^{-1}\right\|_F.$$

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*Corresponding author.
Email address: luisglez@dma.ulpgc.es (Luis González)

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Let us recall that the solution of the linear system

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \;\; x, b \in \mathbb{R}^n \;\; (A \;\text{ nonsingular and sparse}) \tag{1.1}$$

is usually performed by iterative methods based on Krylov subspaces (see, e.g., [1, 2]).

To improve the convergence of these Krylov methods, system (1.1) can be preconditioned with an adequate preconditioning nonsingular matrix N, transforming it into any of the equivalent systems [3]

$$NAx = Nb,$$
$$ANy = b, \qquad x = Ny,$$

the so-called left and right preconditioned systems, respectively. In this paper, we address only the case of the right-hand side preconditioned matrices AN (analogous results can be obtained for the left-hand side preconditioned matrices NA).

Often, the preconditioning of system (1.1) is performed in order to get a preconditioned matrix AN as close as possible to the identity in some sense. The preconditioner N is called an approximate inverse of A. The closeness of AN to I may be measured by using a suitable matrix norm like, for instance, the Frobenius norm [3]. In this way, the best preconditioner N (with respect to the Frobenius norm) of system (1.1) in an arbitrary subspace S of $\mathbb{R}^{n\times n}$ can be obtained by minimizing the residual Frobenius norm $\|AM - I\|_F$ over the subspace S. That is, the best preconditioner N of system (1.1) in subspace S is the solution to the minimization problem; see, e.g., [4]

$$\min_{M\in S} \|AM - I\|_F = \|AN - I\|_F \,. \tag{1.2}$$

Although some of the results presented in this paper are also valid for the case that the solution N to problem (1.2) is singular, from now on, we assume that matrix N (and thus also matrix AN) is a nonsingular matrix.

Taking advantage of the Frobenius inner product in $\mathbb{R}^{n\times n}$, the matrix AN defined by Eq. (1.2) can be obtained as the orthogonal projection of the identity onto the subspace AS. The solution N to problem (1.2) will be referred to as the "optimal" preconditioner of system (1.1) (or as the "best" approximate inverse of matrix A) in the subspace S. In the following, the preconditioning of a linear system with the optimal preconditioner N defined by problem (1.2)), will be referred to as the "optimal preconditioning" of system (1.1) in subspace S. In [5], the solution N to problem (1.2) is called the S-Moore-Penrose inverse of matrix A, and it is theoretically analyzed as a natural generalization of the classical Moore-Penrose inverse.

We must highlight here that the purpose of this paper is purely theoretical, and the relation between problem (1.2) and the preconditioning problem is also analyzed here from a theoretical point of view, and not looking for numerical or computational immediate approaches. In particular, the terms "optimal preconditioner" or 'best approximate inverse" are used in the sense of formula (1.2), and not in any other sense of these expressions.

Regarding the above mentioned connection between the Frobenius norm and the art of preconditioning, in [6] the authors consider the following equality

$$\|AQ - I\|_F^2 = \left(\sqrt{n} - \|AQ\|_F\right)^2 + 2\sqrt{n}\,\|AQ\|_F\left(1 - \cos\left(AQ, I\right)\right)$$

to present an interesting geometrical analysis of the practical difficulties for building accurate approximate inverses Q with a prescribed sparsity pattern.

Moreover, in [6, 7, 8], some geometrical properties and bounds on the Frobenius condition number for positive definite matrices are derived. In this paper we address the analysis of the normalized Frobenius condition number to a different case, namely for the orthogonal projections AN of the identity onto the matrix subspaces AS.

The following are the main goals and the organization of this paper. First, in Section 2, we provide some inequalities for the normalized Frobenius condition number $\widehat{\kappa}_F\left(AN\right)$ of matrix AN. Second, in Section 3, a new index $\widehat{\kappa}_F\left(A, S\right)$ is introduced as a natural generalization of the normalized Frobenius condition number $\widehat{\kappa}_F\left(A\right)$ of matrix A. The new index $\widehat{\kappa}_F\left(A, S\right)$ (referred to as the S-normalized Frobenius condition number of A) is closely related to the optimal preconditioner N in the subspace S, and it is compared with $\|AN - I\|_F$. Finally, some concluding remarks are given in Section 4.

2. Normalized Frobenius condition number of matrix AN

In this section, we present some upper and lower bounds on the normalized Frobenius condition number $\widehat{\kappa}_F\left(AN\right)$ of the orthogonal projection AN defined by Eq. (1.2).

The fact that the Frobenius condition number of the $n \times n$ identity matrix I is $\kappa_F\left(I\right) = n$, implies that, with respect to this condition number, the identity matrix becomes more and more ill-conditioned as n increases (i.e., $\kappa_F(I) \to \infty$ as $n \to \infty$). This makes the classical Frobenius condition number $\kappa_F(\cdot)$ inadequate as a measure of the conditioning of a linear system of equations [9].

For this reason, instead of using the classical Frobenius condition number $\kappa_F(\cdot)$, throughout this paper we use a more meaningful measure, based on the normalized Frobenius norm

$$\frac{1}{\sqrt{n}} \|M\|_F = \sqrt{\frac{1}{n} tr\left(MM^T\right)},$$

and henceforth referred to as the normalized Frobenius condition number. This normalized measure of conditioning is denoted by $\widehat{\kappa}_F(\cdot)$, and defined for all nonsingular $n \times n$ real matrix M as

$$\widehat{\kappa}_F(M) = \frac{1}{n} \|M\|_F \left\|M^{-1}\right\|_F = \frac{1}{n} \kappa_F(M). \tag{2.1}$$

Now, from Eq. (2.1) it is obvious that $\widehat{\kappa}_F(I) = 1$, and also that

$$\widehat{\kappa}_F(M) = \frac{1}{n} \|M\|_F \left\|M^{-1}\right\|_F \geq \frac{1}{n} \left\|MM^{-1}\right\|_F = \frac{1}{n} \|I\|_F = \frac{\sqrt{n}}{n},$$

i.e., for all nonsingular matrix $M \in \mathbb{R}^{n\times n}$, we have

$$\widehat{\kappa}_F(M) \geq \frac{\sqrt{n}}{n}. \tag{2.2}$$

The following lemma provides us with lower and upper bounds on the normalized Frobenius condition number $\widehat{\kappa}_F(AN)$ of the orthogonal projection AN, involving its largest and its smallest singular value. From now on, we denote by $\{\sigma_i\}_{i=1}^n$ the set of singular values of matrix AN arranged, as usual, in nonincreasing order, i.e.,

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be the solution to problem* (1.2)*. Then*

$$\frac{\sigma_1}{n} \leq \frac{\sigma_1}{n\sigma_n} \leq \widehat{\kappa}_F(AN) \leq \frac{\sigma_1}{\sigma_n} < \frac{\sqrt{n}}{\sigma_n}.$$

Proof. Denote by $\|\cdot\|_2$ and by $\kappa_2(\cdot)$ the spectral matrix norm and the spectral condition number, respectively. Using the well-known relations between the spectral and the Frobenius matrix norms [10]

$$\|\cdot\|_2 \leq \|\cdot\|_F \leq \sqrt{n}\,\|\cdot\|_2\,,$$

we get

$$\frac{1}{n}\kappa_2(AN) \leq \widehat{\kappa}_F(AN) \leq \kappa_2(AN)\,,$$

i.e.,

$$\frac{\sigma_1}{n\sigma_n} \leq \widehat{\kappa}_F(AN) \leq \frac{\sigma_1}{\sigma_n}.$$

Now, on one hand, taking into account the following property of the orthogonal projection AN; see, e.g., [4, 11]

$$0 \leq \|AN\|_F^2 = tr(AN) \leq n, \tag{2.3}$$

we get

$$\sigma_1^2 < \sum_{i=1}^{n} \sigma_i^2 = \|AN\|_F^2 \leq n \quad \Rightarrow \quad \sigma_1 < \sqrt{n}.$$

On the other hand, we use the following fact derived in [11]: The smallest singular value of the orthogonal projection AN of the identity onto the subspace AS is never greater than 1, i.e.,

$$0 < \sigma_n \leq 1.$$

Hence, we get

$$\frac{\sigma_1}{n} \leq \frac{\sigma_1}{n\sigma_n} \leq \widehat{\kappa}_F(AN) \leq \frac{\sigma_1}{\sigma_n} < \frac{\sqrt{n}}{\sigma_n}. \qquad \square$$

The following lemma provides lower and upper bounds on the normalized Frobenius condition number $\widehat{\kappa}_F(AN)$ of the orthogonal projection AN, involving the Frobenius norms of both matrix AN and its inverse.

Lemma 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be the solution to problem* (1.2)*. Then*

$$\frac{1}{n}\|AN\|_F \leq \widehat{\kappa}_F(AN) \leq \frac{\sqrt{n}}{n}\left\|(AN)^{-1}\right\|_F.$$

Proof. To prove the left-hand inequality, it suffices to use Eqs. (2.2) and (2.3)

$$\widehat{\kappa}_F(AN) \geq \frac{\sqrt{n}}{n} \geq \frac{\|AN\|_F}{n}.$$

To prove the right-hand inequality, we use again Eq. (2.3)

$$\widehat{\kappa}_F(AN) = \frac{1}{n}\|AN\|_F \left\|(AN)^{-1}\right\|_F \leq \frac{\sqrt{n}}{n}\left\|(AN)^{-1}\right\|_F. \qquad \square$$

Remark 2.1. By the way, from the right-hand side inequality in Lemma 2.2 and from Eq. (2.2) we derive the following relationship between the Frobenius norms of the inverses of matrices A and its best approximate inverse N.

$$\left\|A^{-1}\right\|_F \left\|N^{-1}\right\|_F \geq \left\|(AN)^{-1}\right\|_F \geq \sqrt{n}\,\widehat{\kappa}_F(AN) \geq 1 \Rightarrow \left\|A^{-1}\right\|_F \left\|N^{-1}\right\|_F \geq 1.$$

3. The index $\widehat{\kappa}_F(A, S)$

In this section, a new index $\widehat{\kappa}_F(A, S)$, closely related to the optimal preconditioning in the subspace S, is defined and compared with the normalized Frobenius condition number of matrix A. For this purpose, our starting point is the following upper bound on the cosine of the angle between the orthogonal projection AN and the identity; see [6, 12] and Eq. (2.3)

$$\begin{aligned}\cos(AN, I) &= \frac{\langle AN, I\rangle_F}{\|AN\|_F \|I\|_F} = \frac{tr(AN)}{\|AN\|_F \sqrt{n}} = \frac{\sqrt{tr(AN)}}{\sqrt{n}} \\ &= \frac{\|AN\|_F}{\sqrt{n}} \leq \frac{\|A\|_F \|N\|_F}{\sqrt{n}} = \frac{\frac{1}{n}\|A\|_F \|N\|_F}{\sqrt{n}/n}. \end{aligned} \tag{3.1}$$

This suggests the following definition.

Definition 3.1. Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be solution to problem (1.2). Then the S-normalized Frobenius condition number of matrix A is defined by

$$\widehat{\kappa}_F(A, S) = \frac{1}{n}\|A\|_F \|N\|_F.$$

The S-normalized Frobenius condition number $\widehat{\kappa}_F(A,S)$ of matrix A, can be seen as *the optimal (right) approximate normalized Frobenius condition number of matrix A in S*, since matrix N is the optimal (right) approximate inverse of matrix A in subspace S.

Obviously, the S-normalized Frobenius condition number of matrix A generalizes its normalized Frobenius condition number, that is (see Eq. (2.1)),

$$S \ni A^{-1} \Rightarrow N = A^{-1} \Rightarrow \widehat{\kappa}_F(A,S) = \frac{1}{n}\|A\|_F\|N\|_F$$
$$= \frac{1}{n}\|A\|_F\left\|A^{-1}\right\|_F = \widehat{\kappa}_F(A),$$

e.g.,

$$\widehat{\kappa}_F\left(A,\mathbb{R}^{n\times n}\right) = \widehat{\kappa}_F\left(A, span\left\{A^{-1}\right\}\right) = \widehat{\kappa}_F(A).$$

Remark 3.1. Note that Definition 3.1 can be extended to any complex matrix $A \in \mathbb{C}^{m\times n}$, by considering a subspace $S \subset \mathbb{C}^{n\times m}$ and, in fact, obtaining right and left S-normalized Frobenius condition numbers, respectively associated to the solutions N and N' of the minimization problems

$$\min_{M\in S}\|AM - I_m\|_F = \|AN - I_m\|_F, \qquad \min_{M\in S}\|MA - I_n\|_F = \|N'A - I_n\|_F.$$

However, as already mentioned, in this paper we restrict our study (and thus Definition 3.1) to the case of the (right) minimization problem (1.2), associated to the right preconditioning matrix N of a linear system $Ax = b$ ($A \in \mathbb{R}^{n\times n}$, A nonsingular).

3.1. Lower bounds on $\widehat{\kappa}_F(A,S)$

The following theorem provides different lower bounds on $\widehat{\kappa}_F(A,S)$.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be the solution to problem* (1.2)*. Then*
(i) $\frac{1}{n}\left|\langle A,N\rangle_F\right| \leq \widehat{\kappa}_F(A,S)$.

(ii) $\frac{\sqrt{n}}{n}\cos(AN,I) = \frac{\sqrt{tr(AN)}}{n} = \frac{\|AN\|_F}{n} \leq \widehat{\kappa}_F(A,S)$.

(iii) $\frac{tr(AN)}{n\sqrt{n}} \leq \widehat{\kappa}_F(A,S)$.

(iv) $\frac{2}{n^2}tr(A)\,tr(N) - \frac{1}{n}\left|\langle A,N\rangle_F\right| \leq \widehat{\kappa}_F(A,S)$.

(v) $\frac{tr(A)tr(N)}{n^2} \leq \widehat{\kappa}_F\left(A,S\right)$.

(vi) $\frac{1}{\|A^{-1}\|_F\|N^{-1}\|_F} \leq \widehat{\kappa}_F\left(A,S\right)$.

Proof.

(i) It suffices to use the Cauchy-Schwarz inequality.

(ii) Using Eq. (3.1), the proof is straightforward.

(iii) It suffices to use Eq. (3.1)

$$1 \geq \cos\left(AN, I\right) = \frac{tr\left(AN\right)}{\|AN\|_F\sqrt{n}} \geq \frac{tr\left(AN\right)}{\|A\|_F\|N\|_F\sqrt{n}} = \frac{tr\left(AN\right)}{\widehat{\kappa}_F\left(A,S\right)n\sqrt{n}}.$$

(iv) Using the well-known Buzano's inequality (an extension of the Cauchy-Schwarz inequality in an inner product space)

$$|\langle a,x\rangle \cdot \langle x,b\rangle| \leq \frac{1}{2}\left(\|a\|\,\|b\| + |\langle a,b\rangle|\right)\|x\|^2$$

for the Frobenius inner product and for $a = A,\ x = I,\ b = N$, we get

$$\begin{aligned} tr\left(A\right)tr\left(N\right) &= \langle A,I\rangle_F \cdot \langle N,I\rangle_F \leq |\langle A,I\rangle_F \cdot \langle I,N\rangle_F| \\ &\leq \frac{1}{2}\left(\|A\|_F\|N\|_F + |\langle A,N\rangle_F|\right)\|I\|_F^2 \\ &= \frac{n}{2}\left(\|A\|_F\|N\|_F + |\langle A,N\rangle_F|\right) \\ &= \frac{n^2}{2}\left(\widehat{\kappa}_F\left(A,S\right) + \frac{1}{n}|\langle A,N\rangle_F|\right). \end{aligned}$$

(v) Using (iv) and (i), we get

$$\begin{aligned} \widehat{\kappa}_F\left(A,S\right) &\geq \frac{2}{n^2}tr\left(A\right)tr\left(N\right) - \frac{1}{n}|\langle A,N\rangle_F| \geq \frac{2}{n^2}tr\left(A\right)tr\left(N\right) - \widehat{\kappa}_F\left(A,S\right) \\ &\Rightarrow \widehat{\kappa}_F\left(A,S\right) \geq \frac{tr\left(A\right)tr\left(N\right)}{n^2}. \end{aligned}$$

(vi) Using Eq. (2.2), we get

$$\begin{aligned} \widehat{\kappa}_F\left(A,S\right)\left\|A^{-1}\right\|_F\left\|N^{-1}\right\|_F &= \frac{1}{n}\|A\|_F\|N\|_F\left\|A^{-1}\right\|_F\left\|N^{-1}\right\|_F \\ &= n\left(\frac{1}{n}\|A\|_F\left\|A^{-1}\right\|_F\right)\left(\frac{1}{n}\|N\|_F\left\|N^{-1}\right\|_F\right) \\ &= n\,\widehat{\kappa}_F\left(A\right)\widehat{\kappa}_F\left(N\right) \geq n\frac{\sqrt{n}}{n}\frac{\sqrt{n}}{n} = 1. \qquad \square \end{aligned}$$

The following two corollaries give lower bounds on $\widehat{\kappa}_F(A,S)$ for special cases of matrices A and N.

Corollary 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be the solution to problem* (1.2). *Suppose that matrix A or matrix N (or both) are symmetric. Then*
(i) $\frac{\|AN\|_F^2}{n} = \frac{tr(AN)}{n} \leq \widehat{\kappa}_F(A,S)$.

(ii) $\frac{2}{n^2} tr(A)\, tr(N) - 1 \leq \widehat{\kappa}_F(A,S)$.

Proof.
(*i*) Using Theorem 3.1-(i) and Eq. (2.3), we get

$$\begin{aligned}\widehat{\kappa}_F(A,S) &\geq \frac{1}{n}\left|\langle A,N\rangle_F\right| = \frac{1}{n}\left|tr\left(A^TN\right)\right| = \frac{1}{n}\left|tr\left(AN^T\right)\right| \\ &= \frac{1}{n}\left|tr(AN)\right| = \frac{1}{n}tr(AN) = \frac{1}{n}\|AN\|_F^2 .\end{aligned}$$

(*ii*) Using Theorem 3.1-(iv) and Eq. (2.3), we get

$$\begin{aligned}\widehat{\kappa}_F(A,S) &\geq \frac{2}{n^2}tr(A)\,tr(N) - \frac{1}{n}\left|\langle A,N\rangle_F\right| = \frac{2}{n^2}tr(A)\,tr(N) - \frac{1}{n}\left|tr\left(A^TN\right)\right| \\ &= \frac{2}{n^2}tr(A)\,tr(N) - \frac{1}{n}\left|tr\left(AN^T\right)\right| = \frac{2}{n^2}tr(A)\,tr(N) - \frac{1}{n}\left|tr(AN)\right| \\ &= \frac{2}{n^2}tr(A)\,tr(N) - \frac{1}{n}tr(AN) \geq \frac{2}{n^2}tr(A)\,tr(N) - 1. \qquad \square\end{aligned}$$

Corollary 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be the solution to problem* (1.2). *Suppose that matrix AN is symmetric and positive definite. Then*

$$\frac{1}{n} \leq \widehat{\kappa}_F(A,S).$$

Proof. The proof is straightforward using Theorem 3.1-(ii) and the fact that since the orthogonal projection AN is symmetric and positive definite then $\cos(AN,I) \geq \frac{1}{\sqrt{n}}$; see [12]. $\square$

3.2. Upper bounds on $\widehat{\kappa}_F(A,S)$

Next theorem provides different upper bounds on $\widehat{\kappa}_F(A,S)$, comparing it with the normalized Frobenius condition numbers of matrices A and N.

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Let N be the solution to problem* (1.2). *Then*
(i) $\widehat{\kappa}_F(A,S) \leq \sqrt{n}\,\widehat{\kappa}_F(A)$.

(ii) $\widehat{\kappa}_F(A,S) \leq \sqrt{n}\,\widehat{\kappa}_F(N)$.

(iii) $\widehat{\kappa}_F(A,S) \leq n \cdot \min\left\{\widehat{\kappa}_F^2(A), \widehat{\kappa}_F(A)\,\widehat{\kappa}_F(N), \widehat{\kappa}_F^2(N)\right\}$.

Proof.
(i) Using Eqs. (2.1) and (2.3), we get

$$\begin{aligned}\widehat{\kappa}_F(A,S) &= \frac{1}{n}\|A\|_F\|N\|_F = \frac{1}{n}\|A\|_F\left\|A^{-1}(AN)\right\|_F \\ &\leq \frac{1}{n}\|A\|_F\left\|A^{-1}\right\|_F\|AN\|_F = \|AN\|_F\,\widehat{\kappa}_F(A) \leq \sqrt{n}\,\widehat{\kappa}_F(A).\end{aligned}$$

(ii) Using again Eqs. (2.1) and (2.3), we get

$$\begin{aligned}\widehat{\kappa}_F(A,S) &= \frac{1}{n}\|A\|_F\|N\|_F = \frac{1}{n}\left\|(AN)\,N^{-1}\right\|_F\|N\|_F \\ &\leq \frac{1}{n}\|AN\|_F\|N\|_F\left\|N^{-1}\right\|_F = \|AN\|_F\,\widehat{\kappa}_F(N) \leq \sqrt{n}\,\widehat{\kappa}_F(N).\end{aligned}$$

(iii) It suffices to use (i), (ii) and Eq. (2.2). $\square$

Remark 3.2. It is important to highlight that the exact computation of the normalized Frobenius condition number $\widehat{\kappa}_F(A)$ of matrix A is, in general, not feasible since it requires to compute the Frobenius norm of the unknown inverse of A; see Eq. (2.1). On the contrary, the S-normalized Frobenius condition number $\widehat{\kappa}_F(A,S)$ of matrix A, introduced in this paper, is an explicitly computable quantity. Indeed, it can be easily computed simply by evaluating the Frobenius norms of both matrices A and N; see Definition 3.1. Regarding the optimal approximate inverse N of A over the subspace S, let us mention that this matrix can be explicitly computed, for instance from an orthogonal basis of subspace AS. Such orthogonal basis can be easily obtained from a basis of subspace S and using the Gram-Schmidt orthogonalization procedure if necessary [4].

4. Conclusions

In this paper, we have considered the optimal approximate inverse N (in the Frobenius sense) for a given nonsingular matrix $A \in \mathbb{R}^{n\times n}$, among all matrices belonging to a fixed matrix subspace $S \subset \mathbb{R}^{n\times n}$, that is, the solution N to problem (1.2). Then, in the theoretical context of the preconditioning problem for large linear systems, we have focused on the optimal preconditioned matrix AN, that is, the orthogonal projection (with respect to the Frobenius inner product) of the identity matrix onto the subspace AS.

We have derived some inequalities for the normalized Frobenius condition number $\widehat{\kappa}_F(AN)$ of matrix AN. In addition, we have introduced a new index, closely related to the optimal preconditioning of large linear systems, using Frobenius norm minimization: The so-called S-normalized Frobenius condition number of matrix A, denoted by $\widehat{\kappa}_F(A,S)$. Different lower and upper bounds on $\widehat{\kappa}_F(A,S)$ have been obtained. The new index $\widehat{\kappa}_F(A,S)$ generalizes the classical normalized Frobenius condition number $\widehat{\kappa}_F(A)$ of matrix A, and both numbers coincide when $N = A^{-1}$ (i.e., when $S \ni A^{-1}$). In the general case, when $A^{-1} \notin S$, $\widehat{\kappa}_F(A,S)$ provides the best approximation in the subspace S to $\widehat{\kappa}_F(A)$. The latter can not be computed in general; the former can always be exactly computed from a basis of subspace S.

Finally, for future research, one can think on the possibility of establishing some relations between the minimum residual Frobenius norm $\|AN - I\|_F$ and the new proposed index $\widehat{\kappa}_F(A,S)$, in order to estimate the quality of the approximation $N \approx A^{-1}$ in terms of the value of $\widehat{\kappa}_F(A,S)$ (which can be exactly computed).

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