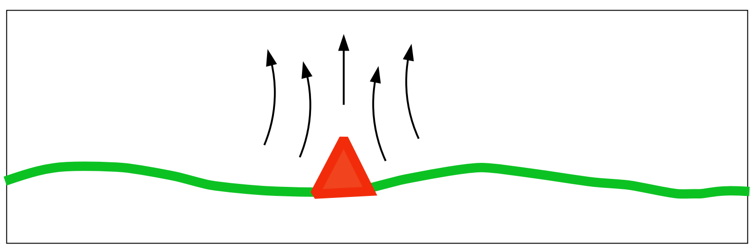
Modelling of convection in forest fires

M.I.Asensio, L. Ferragut
Dpto. Matemática Aplicada, Universidad de Salamanca
J. Simon
CNRS, Laboratoire de Mathématiques Appliquées, Université Blaise Pascal, Aubiére cedex, France

November 2002

Convection Model in Forest Fires

We present a model coupling the fire propagation equations in a bidimensional domain representing the surface, and the air movement equations in a three dimensional domain representing an air layer



As the air layer thickness is small compared with its length, an asymptotic analysis gives a three dimensional convective model governed by a bidimensional equation verified by a stream function.

Combustion Equations

Pyrolysis: Solid fuel, y_s

$$\partial_t y_s = -\beta_s y_s e^{\frac{\vartheta}{1+\varepsilon_s \vartheta}}$$

Convection, Diffusion and Reaction: Gaseous fuel, y_g

$$\partial_t y_g + v \cdot \nabla_x y_g - \kappa \Delta_x y_g = -\beta y_g y_o e^{\frac{\vartheta}{1+\varepsilon_g \vartheta}} + \beta_s y_s e^{\frac{\vartheta}{1+\varepsilon_s \vartheta}} - \alpha_g y_g$$

Convection, Diffusion and Reaction: Oxygen, y_o

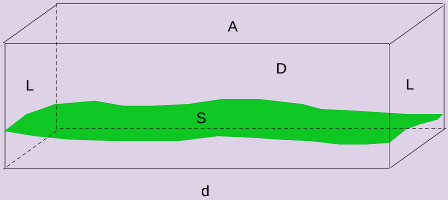
$$\partial_t y_o + v \cdot \nabla_x y_o - \kappa \Delta_x y_o = -\beta y_g y_o e^{\frac{\vartheta}{1+\varepsilon_g \vartheta}}$$

Convection, Diffusion and Reaction: Temperature, ϑ

$$\partial_t \vartheta + v \cdot \nabla_x \vartheta - \nabla_x \cdot (\kappa_r (1+\varepsilon_g \vartheta)^3 \nabla_x \vartheta) = y_g y_o e^{\frac{\vartheta}{1+\varepsilon_g \vartheta}} - \alpha \vartheta$$

where radiation is modelled by a nonlinear diffusion term

Wind model: Vertical diffusion model



$$D = \{(\mathbf{x}, z) : \mathbf{x} \in d,\ \ h(x) < z < \delta\}$$
$$S = \{(\mathbf{x}, z) : \mathbf{x} \in d,\ \ z = h(\mathbf{x})\}$$
$$A = \{(\mathbf{x}, z) : \mathbf{x} \in d,\ \ z = \delta\}$$
$$L = \{(\mathbf{x}, z) : \mathbf{x} \in \partial d,\ \ h(\mathbf{x}) < z < \delta\}$$

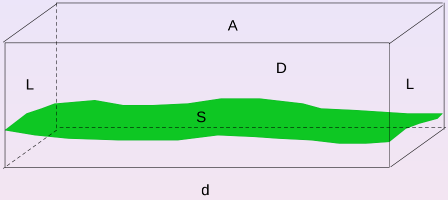
Derivation of the Wind model

The air velocity $\mathbf{U} = (U_1, U_2, U_3)$ and the potential P satisfy the Navier–Stokes equations

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{\mathbf{xz}} \mathbf{U} - \frac{1}{\mathrm{Re}} \Delta_{\mathbf{xz}} U + \nabla_{\mathbf{xz}} P = \varphi Q \mathbf{e}_3$$

- The right-hand side represents the buoyancy forces.
- The density variations due to the temperature have been neglected into the other terms of the equation.
- The air compressibility is also neglected, $\nabla_{\mathbf{xz}} \cdot \mathbf{U} = 0$

Derivation of the Wind model



- On surface S, $\mathbf{U}\cdot\mathbf{N}=0$, $\quad \frac{\partial\mathbf{U}}{\partial\mathbf{N}}|_{tang}=\zeta\mathbf{U}$
- On the air upper boundary A, $\mathbf{U}\cdot\mathbf{N}=0$, $\quad \frac{\partial\mathbf{U}}{\partial\mathbf{N}}|_{tang}=0$
- On the air side boundary L, $\mathbf{U}|_L=(v_m,0)$ where $\partial_z v_m=0$, $\int_{\partial d}(\delta-h)v_m\cdot\mathbf{n}ds=0$

Derivation of the Wind model

- The layer thickness is small in relation to its width $\delta \ll 1$.
- The wind is not too strong $\delta^2 \mathrm{Re} \ll 1$.
- Preserving only the dominant terms and rescaling P, being $\delta = \varphi \mathrm{Re}$

Equations	Boundary conditions
$-\partial^2_{zz}\mathbf{V} + \nabla_{\mathbf{x}} P = 0$	$\partial_z \mathbf{V} = \zeta \mathbf{V}, \quad (\mathbf{V}, W) \cdot \mathbf{N} = 0 \qquad \text{on } S$
$\partial_z P = \varphi Q$	$\partial_z \mathbf{V} = 0, \quad W = 0 \qquad \text{on } A$
$\nabla_{\mathbf{x}} \cdot \mathbf{V} + \partial_z W = 0$	$\mathbf{V} \cdot \mathbf{n} = v_m \cdot \mathbf{n} \qquad \text{on } \partial d$

Derivation of the Wind model

- Defining the horizontal flux at a point $\mathbf{x} \in d$ and time t by
$$\overline{\mathbf{V}}(t, \mathbf{x}) = \int_{h(\mathbf{x})}^{\delta} \mathbf{V}(t, \mathbf{x}, z)\, dz$$
- The incompressibility of $\mathbf{U}$ and the fact that the air does not cross S and A give the incompressibility of the horizontal flux
$$\nabla_{\mathbf{x}} \cdot \overline{\mathbf{V}} = 0$$
- By Stokes $\int_{\partial d} \overline{\mathbf{V}} \cdot nds = \int_d \nabla_{\mathbf{x}} \cdot \overline{\mathbf{V}} dx = 0$ so as $\overline{\mathbf{V}} = (\delta - h)\mathbf{v}_m$ on ∂d we need the hypothesis
$$\int_{\partial d} (\delta - h)\mathbf{v}_m \cdot \mathbf{n} ds = 0$$

Derivation of the Wind model

- The temperature q considered in the combustion equations is obviously the value of Q on the surface, that is
$$q(t,\mathbf{x}) = Q(t,\mathbf{x},h(\mathbf{x}))$$
- We assume that
$$Q(t,\mathbf{x},z) = q(t,\mathbf{x})\frac{\delta - z}{\delta - h(\mathbf{x})}$$
- The velocity $\mathbf{v}$ considered in the combustion equations is the value of $\mathbf{V}$ on the surface, that is, the horizontal component of the wind velocity $\mathbf{U}$ on the surface
$$\mathbf{v}(t,\mathbf{x}) = \mathbf{V}(t,\mathbf{x},h(\mathbf{x}))$$

Derivation of the Wind model

Compute explicitly $P(t, \mathbf{x}, z)$ and $\mathbf{V}(t, \mathbf{x}, z)$ in terms of a 2D potential p.

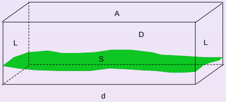
For a fixed $\mathbf{x}$, equation $\partial_z P = \lambda Q$ provides

$$P(t, \mathbf{x}, z) = p(t, \mathbf{x}) + \frac{\lambda q(t, \mathbf{x})}{\delta - h(\mathbf{x})}\left(\delta z - \frac{1}{2} z^2\right)$$

Equation $\partial^2_{zz} \mathbf{V} = \nabla_{\mathbf{x}} P$ together with the boundary conditions, provides

$$V(\mathbf{x}, z) = m(\mathbf{x}, z) \nabla p(\mathbf{x}) + n(\mathbf{x}, z) \nabla \hat{T}(\mathbf{x})$$

Vertical diffusion model



Gives the horizontal wind field in a 3D domain by the expression

$$V(\mathbf{x}, z) = m(\mathbf{x}, z)\nabla p(\mathbf{x}) + n(\mathbf{x}, z)\nabla \hat{T}(\mathbf{x})$$

1. $m(\mathbf{x}, z) = \frac{1}{2}z^2 - \delta z - \frac{1}{2}h^2(\mathbf{x}) + (\delta + \xi)h(\mathbf{x}) - \xi\delta$
2. $n(\mathbf{x}, z) = -\frac{1}{24}z^4 + \frac{1}{6}\delta z^3 - \frac{1}{3}\delta^3 z + \frac{1}{24}h^4(\mathbf{x}) - \dots$
3. $p(\mathbf{x})$ is a potential function

Vertical diffusion model

The potential $p(\mathbf{x})$ satisfies the following boundary problem

$$-\nabla(a\nabla p) = \nabla(b\nabla \hat{T}) \quad \text{in } d$$
$$a\frac{\partial p}{\partial n} = -b\frac{\partial \hat{T}}{\partial \nu} + (\delta - h)v_m.\nu \quad \text{on} \quad \partial d$$

1. $a(\mathbf{x}) = \frac{1}{3}(\delta - h(\mathbf{x}))^2(3\xi + \delta - h(\mathbf{x}))$
2. $b(\mathbf{x}) = \frac{1}{30}(\delta - h(\mathbf{x})^2\Big(2\delta^2(2\delta + 5\xi) - 2\delta(\delta - 5\xi)h(\mathbf{x}) - (3\delta + 5\xi)h^2(\mathbf{x}) + h^3(\mathbf{x})\Big)$

Optimal control problem

Let $v = (\delta - h)v_m\nu$ the flow on the boundary

We formulate the former problem as an optimal control problem

Given N experimental measurements of the wind velocity $V_i,\ i = 1, ..., N$, at N given points $P_i = (\mathbf{x}_i, z_i),\ i = 1, ...N$, we search for the value of $v \in L^2_0(\partial\ d)$ such that the value $V(\mathbf{x}_i, z_i)$ given by the expresion $V(\mathbf{x}, z) = m(\mathbf{x}, z)\nabla p(\mathbf{x}) + n(\mathbf{x}, z)\nabla \hat{T}(\mathbf{x})$ are as close as possible to the experimental values V_i.

Optimal control problem

1. $v \in L^2_0(\partial\omega)$ is the control.
2. $$-\nabla(a\nabla p) = \nabla(b\nabla \hat{T}) \quad \text{in } \omega$$
$$a\frac{\partial p}{\partial n} = -b\frac{\partial \hat{T}}{\partial \nu} + (\delta - h)v_m.\nu \quad \text{on} \quad \partial\, d$$
are the state equations.
3. $J(v) = \frac{1}{2}\sum_i \int_\omega \rho_{\epsilon,(\mathbf{x}-\mathbf{x}_i)}(\mathbf{x})\Big(m(\mathbf{x},z)\nabla p(\mathbf{x}) + n(\mathbf{x},z_i)\nabla q(\mathbf{x}) - V_i\Big)^2 + \frac{\alpha}{2}\int_{\partial\, d} v^2$ is the cost function

Optimal control problem

The optimal control problem is caracterised by

- $$\int_\omega a\nabla p(u)\nabla\varphi + \frac{1}{\alpha}\int_{\partial\omega} q\varphi = -\int_\omega b\nabla\hat{T}\nabla\varphi \quad \forall\varphi\in V$$

- $$\int_\omega a\nabla q(u)\nabla\psi$$
$$-\sum_{i=1}^{N}\int_\omega \rho_\epsilon(x-x_i)(m\nabla p(u)+n\nabla\hat{T}-V_i)m\nabla\psi = 0 \quad \forall\psi\in V$$

- $$u = -\frac{1}{\alpha}q \quad \text{on} \quad \partial\, d$$

4.4. Finite element approximation

Let us discretize the approached equations (42)–(43). Let $\mathcal{T}_H$ be a uniform triangulation of ω corresponding to a discretization parameter H and let V_H be the associated space of P_1 (or P_2) finite elements. Besides a better order of convergence, a reason in favor of P_2 against P_1 is that in practical applications, the variable of physical interest is the wind velocity V which is obtained from the potential p using expression (16), involving derivatives.

Choosing a finite element basis $\{\phi_i\}$ for V_H, we introduce the following matrices

$$G = \left\{ \int_\omega a\, \nabla\phi_r \cdot \nabla\phi_k + \eta \int_{\partial\omega} \phi_r\phi_k \right\}_{r,k},$$

$$C_1 = \left\{ \frac{1}{\alpha} \int_{\partial\omega} \phi_r\phi_k \right\}_{r,k}, \quad C_2 = \left\{ \sum_{i=1}^{N} \int_\omega \rho_{\epsilon,i}\, m^2\, \nabla\phi_r \cdot \nabla\phi_k \right\}_{r,k}$$

and the vectors

$$f_p = \left\{ -\int_\omega b\nabla t \cdot \nabla\phi_r \right\}_r, \quad f_q = \left\{ -\sum_{i=1}^{N} \int_\omega \rho_{\epsilon,i}(n\nabla t - V_i)m \cdot \nabla\phi_r \right\}_r.$$

Then, the discrete problem associated to (42)–(43) is the following linear algebraic system:

$$\begin{bmatrix} G & C_1 \\ -C_2 & G \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} f_p \\ f_q \end{bmatrix} \tag{49}$$

The matrix in (49) being nonsymmetric and very ill-conditioned, most of the standard iterative methods fail to converge or have a very slow convergence (this is the case of GMRES-ILU preconditioned). For moderate number of unknowns we use the state-of-the-art sparse

5. NUMERICAL EXAMPLES

5.1. Example 1: Effect of a topography and of a temperature gradient

In this section we consider the effect of two hills on the wind, as well as the effect of the temperature gradient in a square of 6 by 6 kilometers. The ground height and ground temperature are shown on Figure 1.

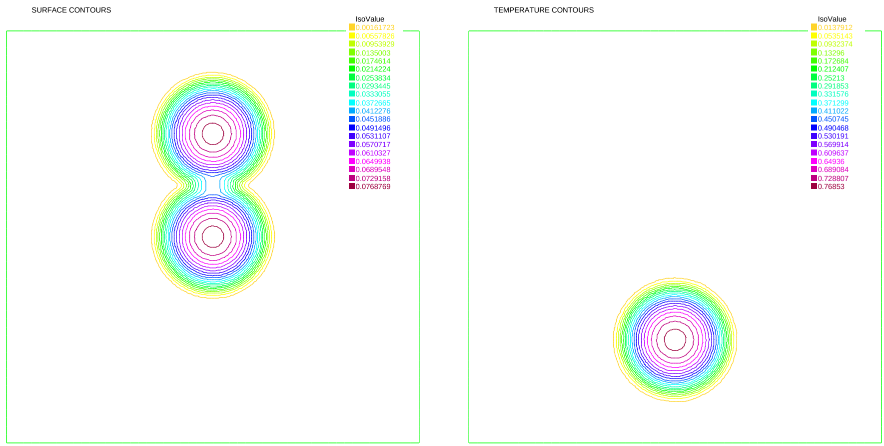


Figure 1. Ground height and ground temperature (Example 1)

The wind velocity is supposed to be known (by experimental measurements) at four points of horizontal coordinates $x = (1., 1.)$, $(5., 1.)$, $(5., 5.)$, $(1., 5.)$ and of height $z = 0.1 + h(x)$, with the same value $V(x,z) = (2., 0.)$ and we take $\alpha = 0.001$.

Figure 2 shows the calculated adjoint state and potential, and Figure 3 shows the calculated velocity module and wind field on the ground surface, that is for $z = h(x)$. As expected, the

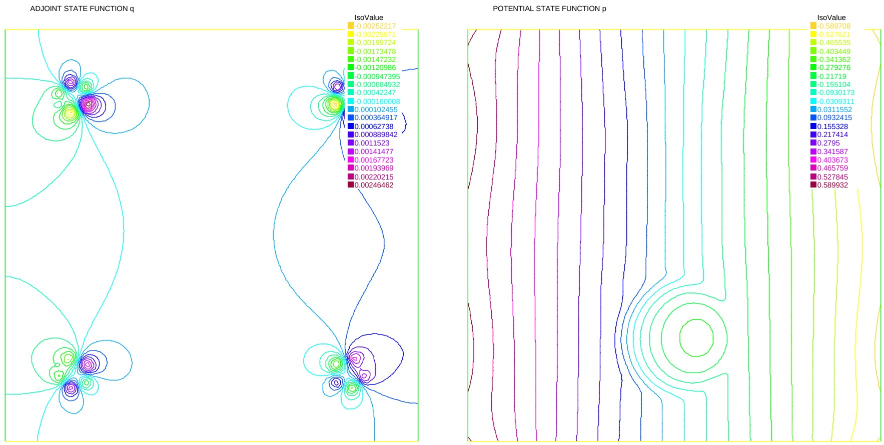


Figure 2. Adjoint state and potential (Example 1)

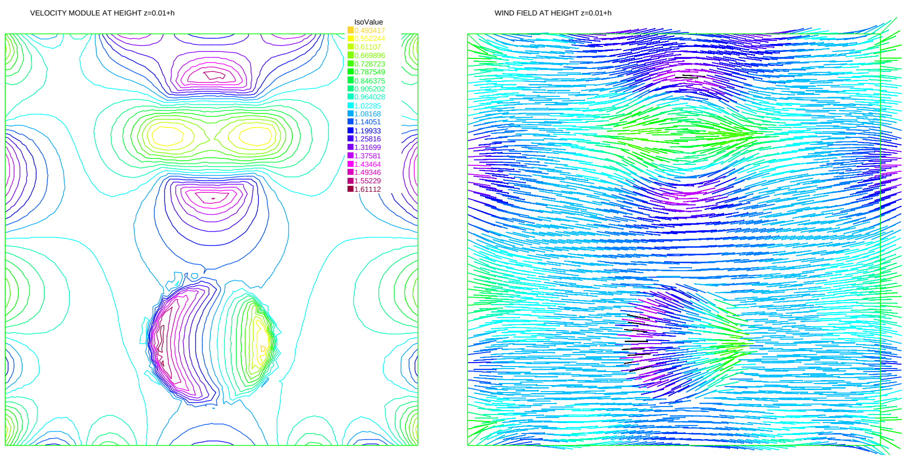


Figure 3. Velocity module and wind field (Example 1)

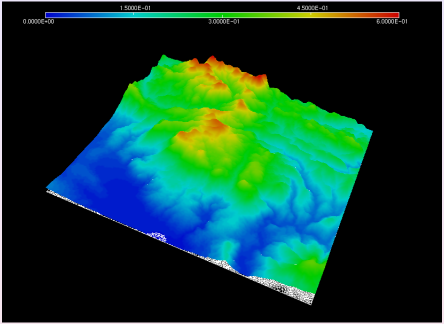


Figure: Surface

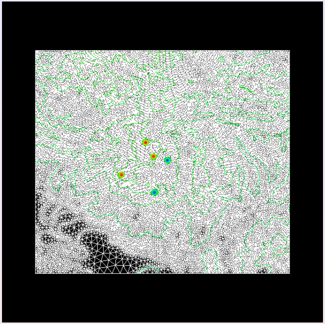


Figure: Position of control points

Functional relationship between GIS data and model parameters

Relationship between rugosity and the friction coefficient of in the wind model

The inverse ξ of the friction coefficient as a function of the rugosity ρ is given by

$$\xi = 0.25(1 + 0.05\rho - 0.01\rho^2)$$

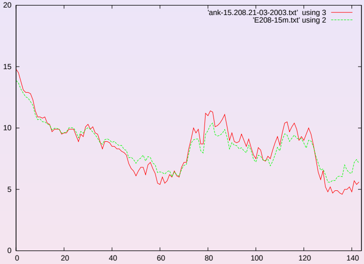


Figure: E208-15m

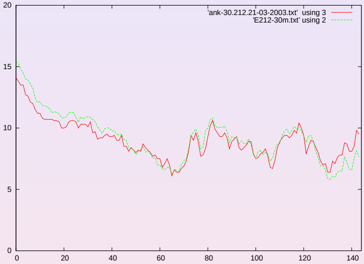


Figure: E212-30m

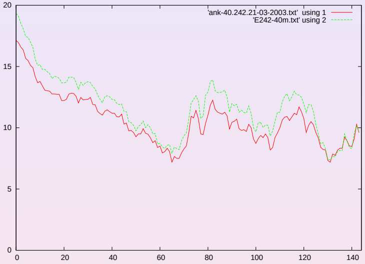


Figure: E242-40m

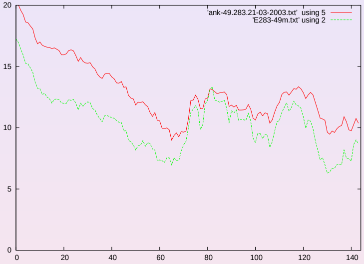


Figure: E283-49m

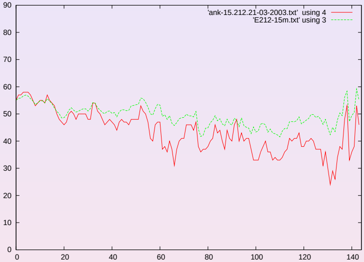


Figure: E212-15m-dir.pdf

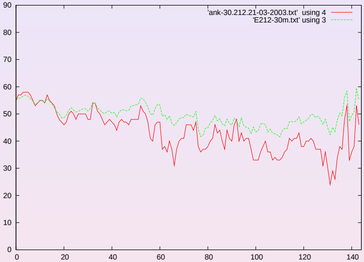
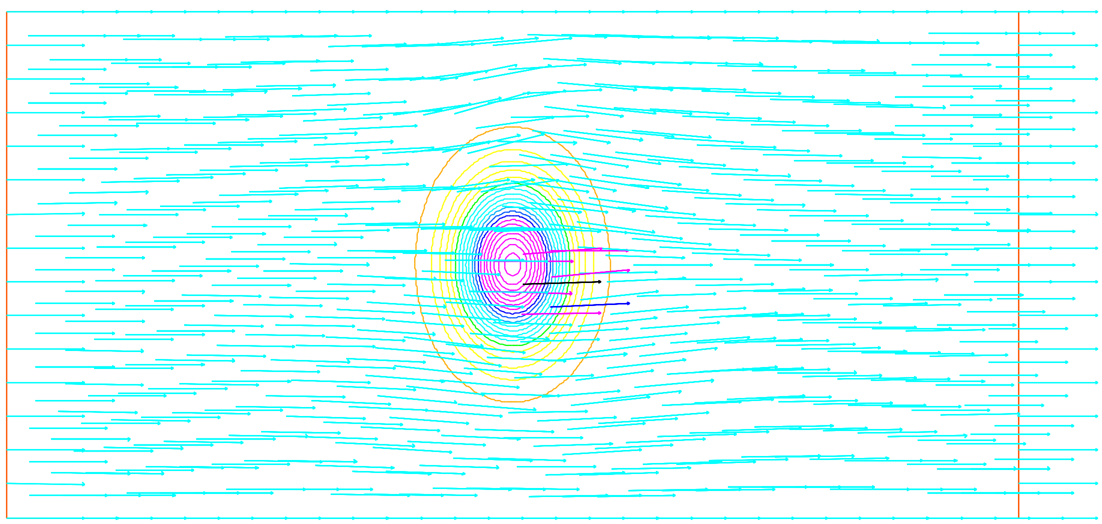


Figure: E212-30m-dir.pdf

Numerical simulation

Domain, contours of the surface and meteorologic wind

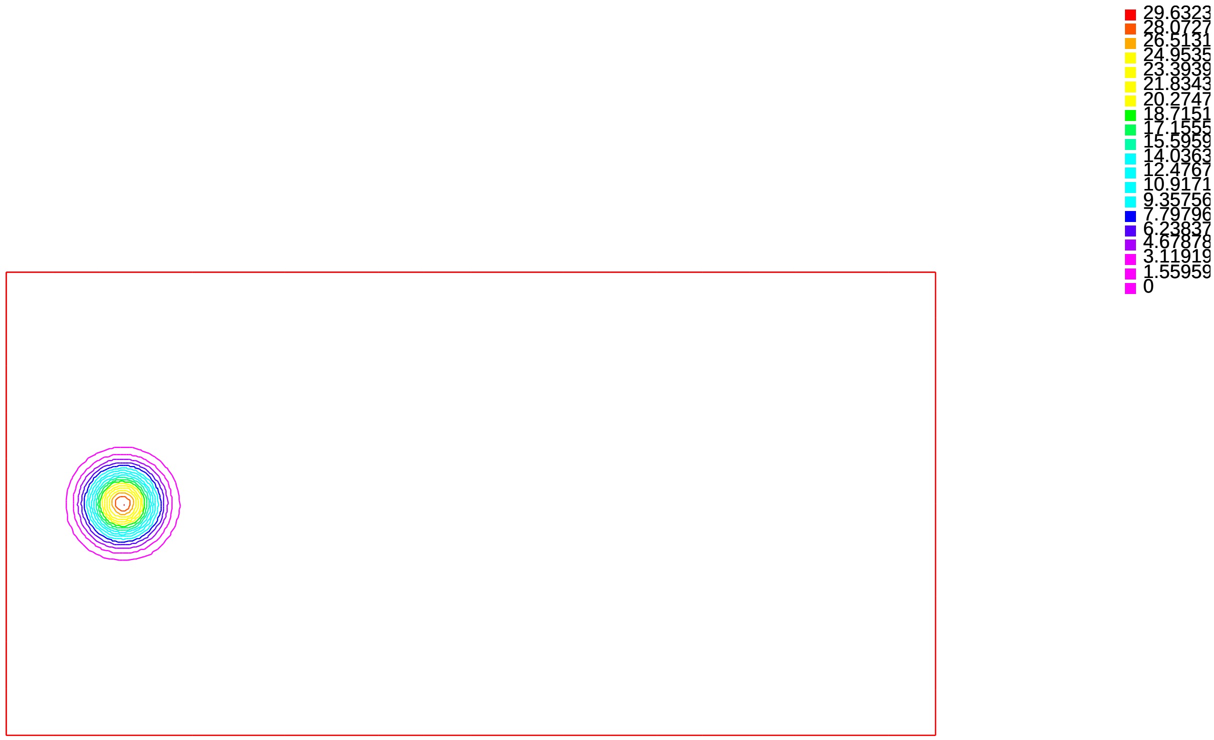


Surface function

$$h(x_1, x_2) = 0.5e^{(-100(x-1.)^2-50(y-0.5)^2)}$$

Numerical simulation

Initial Temperature



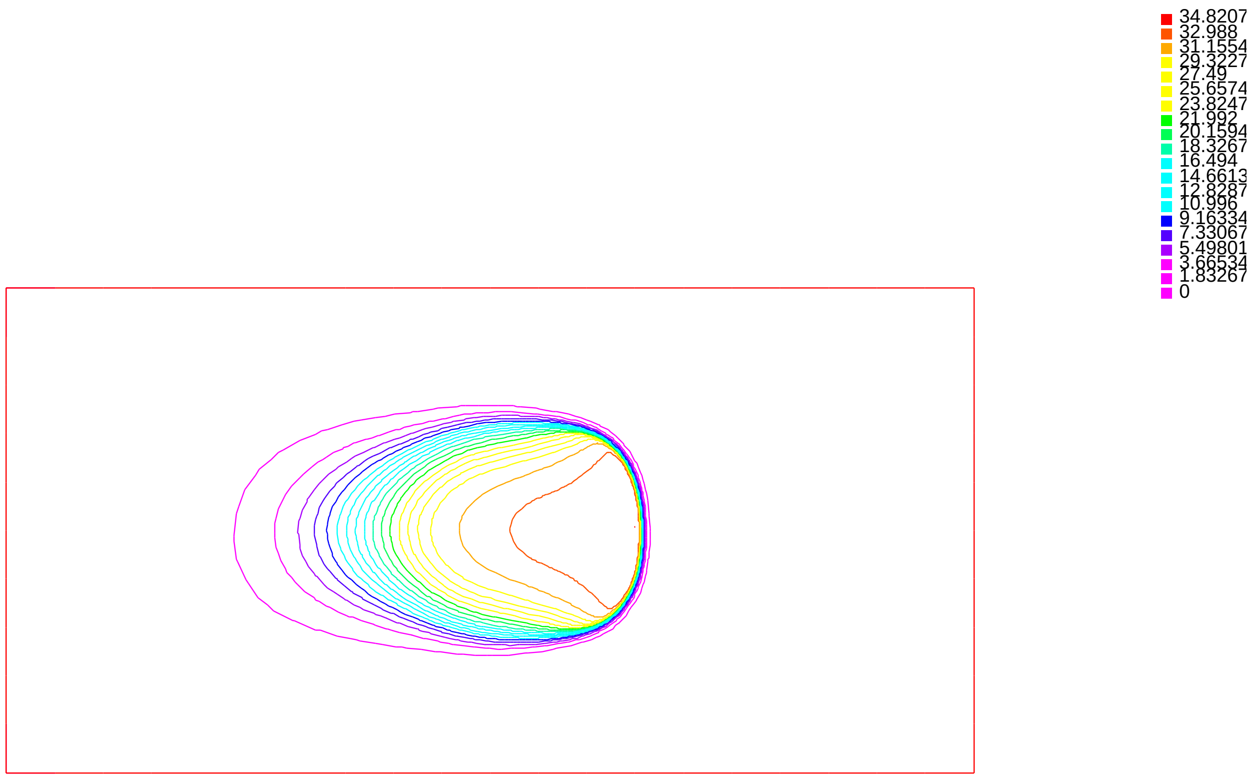
Initial fire focus

$$\vartheta(x_1, x_2) = 30e^{-200(x_1-0.25)^2+(x_2-0.5)^2)}$$

Parameters: $\varepsilon_s = 0.05, \varepsilon_g = 0.04,$
$\beta_s = 1, \beta = 1/1.2, \kappa = 0.1, \kappa_r = 0.1$
$\lambda = 3, \xi = 0.1$

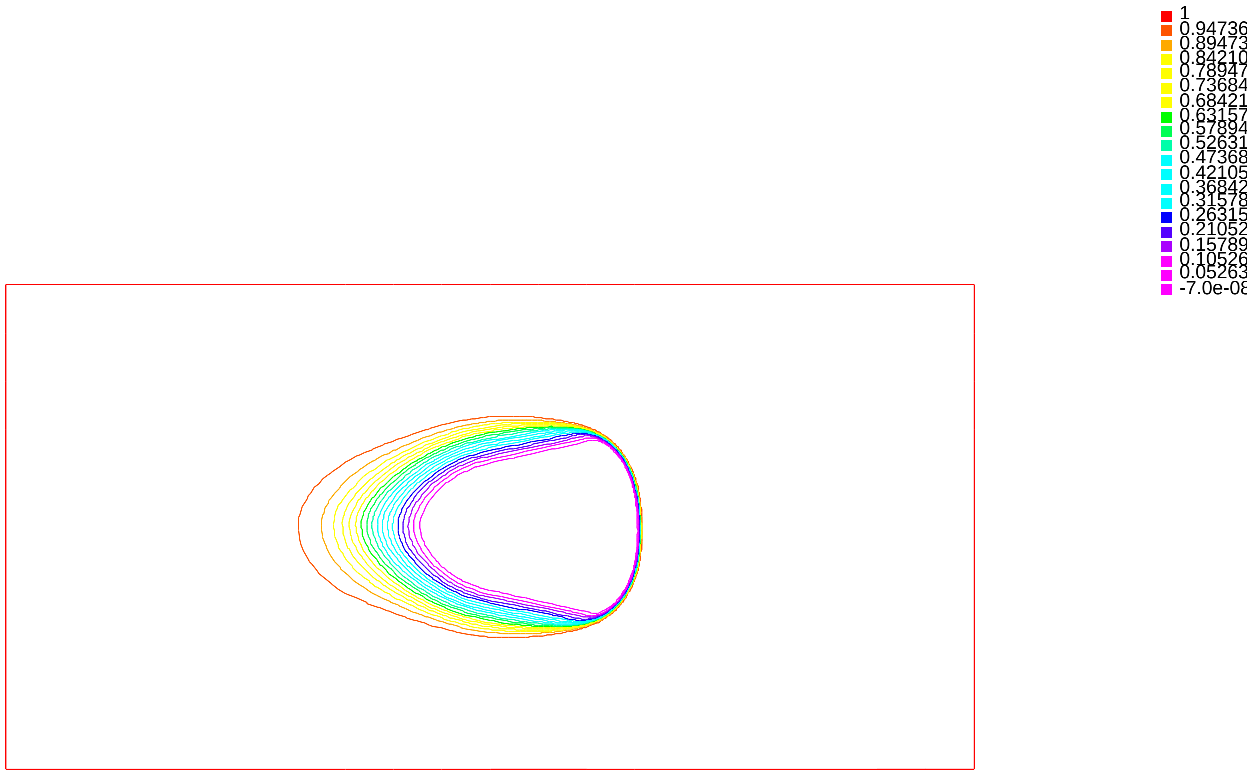
Numerical simulation

Contours of the Temperature after 1000 time steps



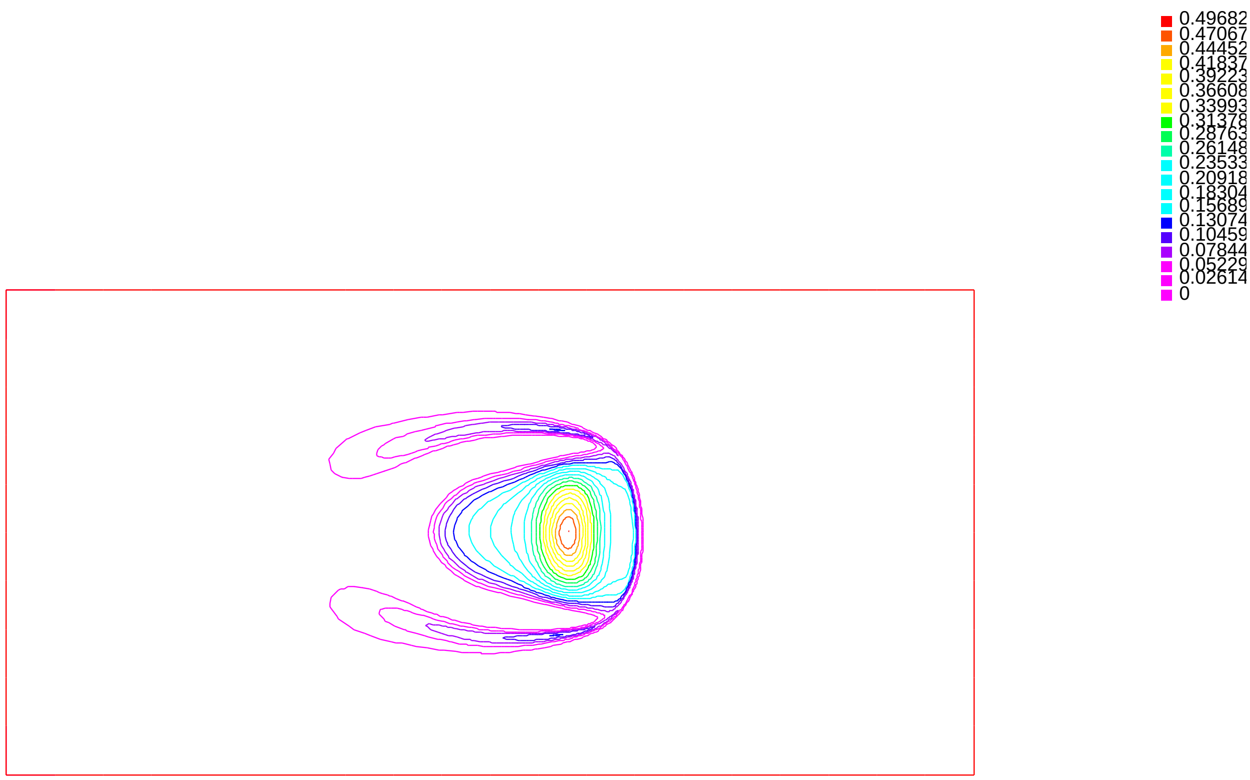
Numerical simulation

Contours of the concentration of Oxygen after 1000 time steps



Numerical simulation

Contours of the concentration of gaseous fuel after 1000 time steps

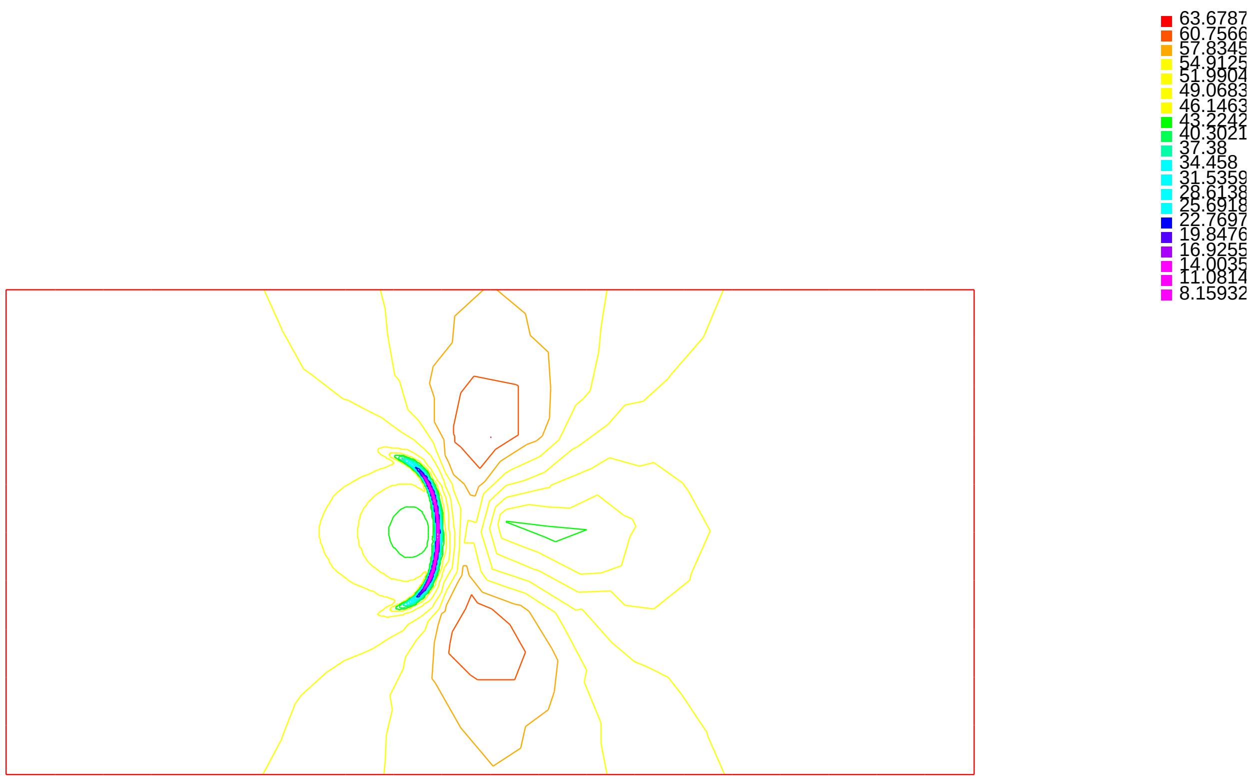


Numerical simulation

Contours of the concentration of solid fuel after 1000 time steps

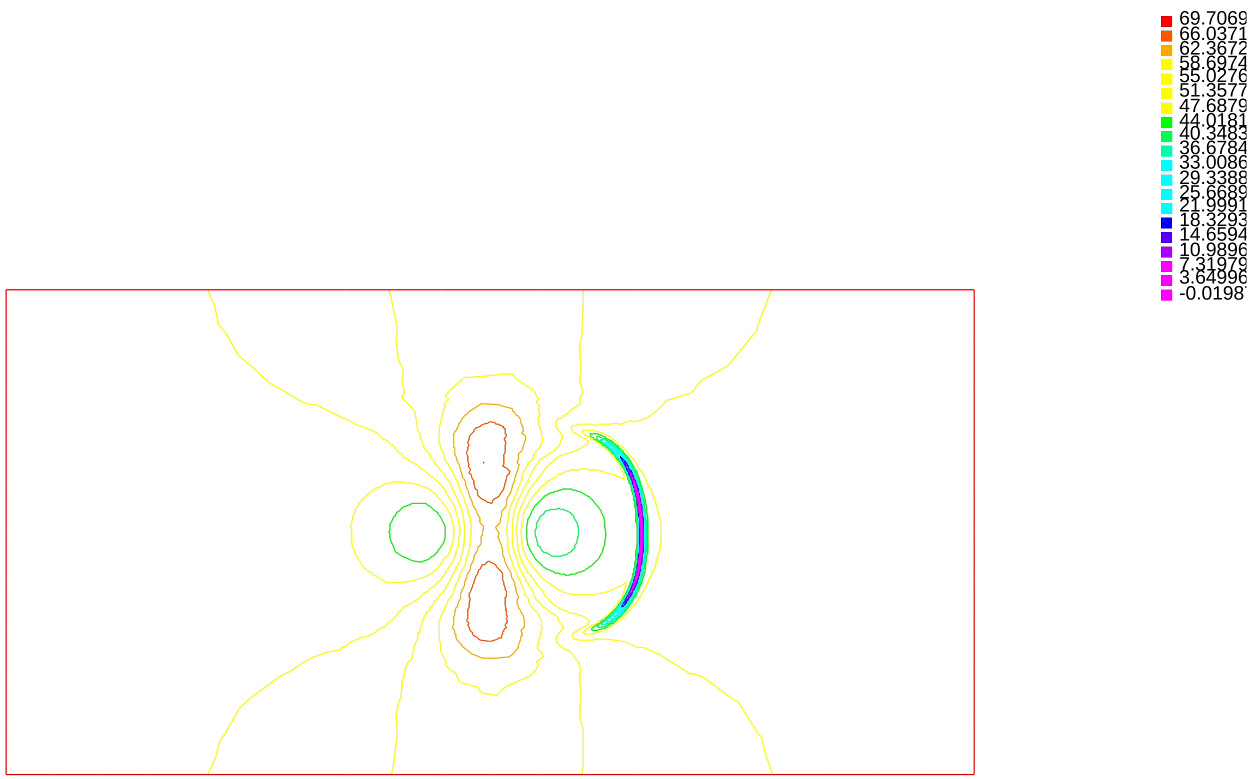
Numerical simulation

Contour plot of the x-component of velocity after 600 time steps



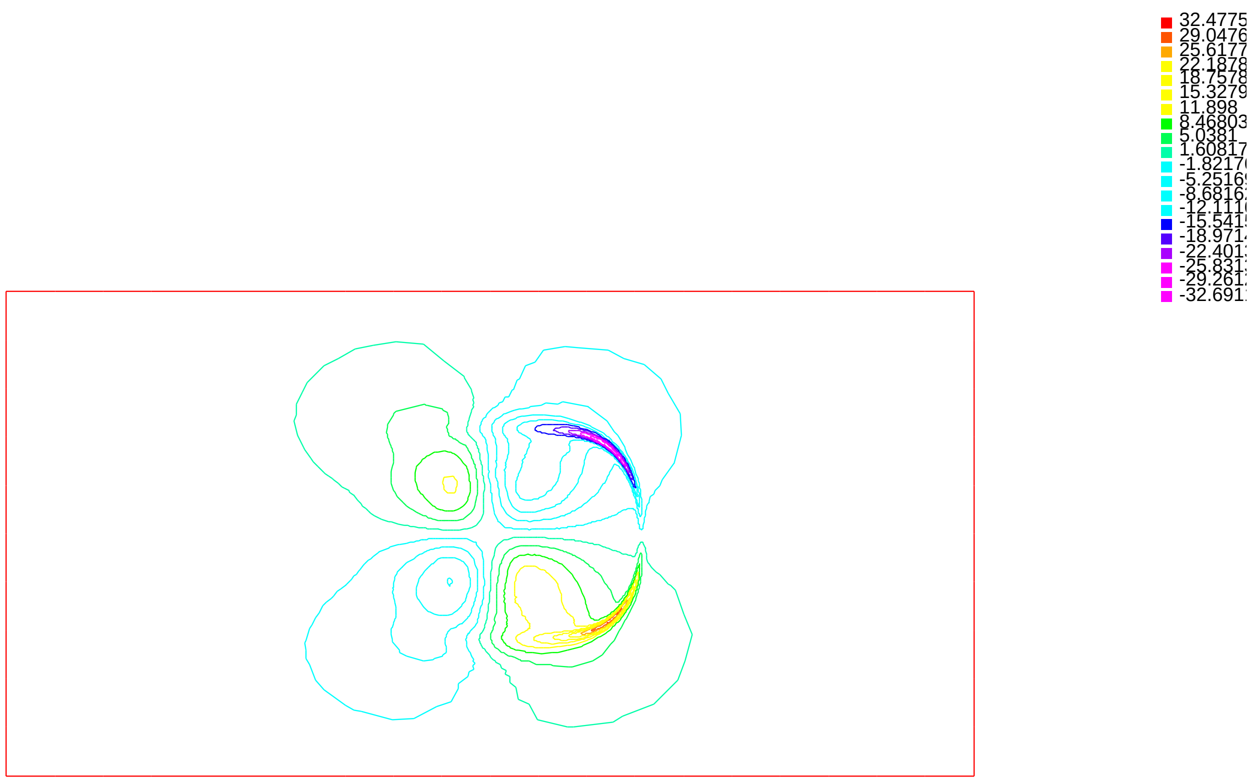
Numerical simulation

Contour plot of the x-component of velocity velocity after 1000 time steps



Numerical simulation

Contour plot of the y-component of velocity after 1000 time steps



Numerical simulation

Finite Element Mesh after 1000 time steps